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Saddlepoint Approximations for the Combined Probability and Joint Probability Density Function of Selected Order Statistics and the Sum of the Remainder

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PREFACE

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A set of N independent random variables $\{x_n\}$ with arbitrary probability density functions $\{p_n(x)\}$ are ordered into a new set of dependent random variables, each with a different probability density function. From this new set, the $M-1$ largest random variables are selected. Then, the sum of the remaining $N+1-M$ random variables is computed, giving a total of M dependent random variables.

Several statistics have been computed for these M random variables, including their joint probability density function and a quantity called the "combined probability and joint probability density function" of particular selections. Most of the results can be expressed in terms of a single Bromwich contour integral in the moment-generating domain. A saddlepoint approximation and first-order correction term are derived for this contour integral and then applied to several typical examples. Actual numerical calculation of these saddlepoint approximations requires evaluation of several functions that have high-order removable singularities, thereby requiring power series expansions near these removable singularities. Expansions with 13 to 14 decimal-digit accuracy have been derived and programmed for this purpose.

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LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS

$A(\lambda)$	Auxiliary function, equation (27)
C	Contour of integration, equation (1)
$c_n(u,\lambda)$	Auxiliary c-function, equation (2)
D	Auxiliary function, equations (39) and (40)
DOF	Degrees of freedom
E	$\exp(z)$, equation (16)
erf	Error integral, equation (18)
G	auxiliary function, equation (23)
$g(k)$	Auxiliary function, equation (84)
k	Index of chi-squared random variable, equation (32)
$L(z)$	Fundamental log function, equations (12), (19), (27), (34)
M	Number of random variables in set under consideration
MGF	Moment-generating function
m_j	Index of j-th largest random variable, equation (1)
n	Index of random variables, equations (2) and (3)
N	Number of random variables, equation (2)
$P(z,\lambda)$	Product function, equation (1)
PDF	Probability density function
$p(u)$	General probability density function, equation (45)
$p_c(u)$	Estimated saddlepoint approximation, equation (53)
$p_n(x)$	Probability density function of $\mathbf{x}_n$, equations (1) and (2)
$p_s(u)$	Saddlepoint approximation, equation (48)
q_M	Combined probability and joint probability density, equation (1)
$r(\lambda)$	Auxiliary function, equation (29)
R_a	Relative error tolerance, equation (50)
R_b	Relative error tolerance, equation (64)
R_1	Relative error, equation (57)
R_2	Relative error, equation (67)
RV	Random variable
SP	Saddlepoint
SPA	Saddlepoint approximation
$U(x)$	Unit step function, equation (1)
X	Set of allowed n values, equation (3)
$\mathbf{x}_n$	n-th random variable
z_m	m-th value of parameter, $m = 1{:}M$, equation (1)
$\mathbf{z}_m$	m-th ordered random variable
ε	Error in saddlepoint location, equation (52)
λ	Variable in moment-generating domain, equation (1)
λ_c	Axis crossing used for estimate, equation (52)

LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS (Cont'd)

λ_s	Saddlepoint location, equation (7)
$\Lambda(\lambda)$	Logarithm of integrand, equation (4)
$\Lambda(\lambda,u)$	Logarithm of general integrand, equation (47)
$\phi(u)$	Normalized Gaussian density function, equations (24) and (26)
$\Phi(t)$	Cumulative Gaussian distribution, equations (25) and (26)
$\mu(\lambda)$	Moment-generating function, equation (45)
$\chi(\lambda)$	Logarithm of moment-generating function, equation (47)
boldface	Random variable
prime	Derivative with respect to λ, equation (5)

SADDLEPOINT APPROXIMATIONS FOR THE COMBINED PROBABILITY AND JOINT PROBABILITY DENSITY FUNCTION OF SELECTED ORDER STATISTICS AND THE SUM OF THE REMAINDER

INTRODUCTION

Detection and location of weak signals in random noise are frequently accomplished by the ordering of the random variables (RVs) in a measured dataset, followed by an investigation of the locations and statistics of several of the largest RVs under consideration. Also of interest are the remaining smaller RVs in the dataset, which can be used to estimate the background noise level and to form a basis for normalization, thereby realizing a constant false alarm processor.

In this report, the original dataset $\{\mathbf{x}_n\}$ is composed of N independent RVs with arbitrary probability density functions (PDFs) $\{p_n(x)\}$. This dataset is ordered into another dataset of dependent RVs, each with a different PDF. From this latter set, the $M-1$ largest RVs are selected. The sum of the remaining $N+1-M$ RVs is then computed, giving a total of M dependent RVs. The joint M-th order PDF of these M dependent RVs is one of the quantities of interest.

For convenience, in this report, the largest RV in set $\{\mathbf{x}_n\}$ is denoted by $\mathbf{z}_1$, the second-largest by $\mathbf{z}_2, \ldots,$ the M-1 largest by $\mathbf{z}_{M-1}$, and the sum of the remaining RVs by $\mathbf{z}_M$. Thus, the first $M-1$ RVs satisfy the restrictions that $\mathbf{z}_1 \geq \mathbf{z}_2 \geq \cdots \geq \mathbf{z}_{M-1}$, while the last RV satisfies the restriction that $\mathbf{z}_M \leq (N+1-M)\,\mathbf{z}_{M-1}$. In two earlier studies, detailed in references 1 and 2, the combined probability and joint PDF of these M RVs was derived in a form involving a one-dimensional Bromwich contour integral in the moment-generating function (MGF) domain. Here, a saddlepoint approximation (SPA) to that contour integral is derived and applied to several typical sets of RVs. In addition, the first-order correction term to the SPA is developed. This development is based heavily on the results and notations of references 1 and 2.

SADDLEPOINT APPROXIMATION FOR COMBINED PROBABILITY AND JOINT PROBABILITY DENSITY FUNCTION

The combined probability and joint PDF of interest is given in reference 2, equation (51), namely,

$$q_M(m_1,\ldots,m_{M-1};z_1,\ldots,z_M) = \prod_{m=1}^{M-2}\{U(z_m - z_{m+1})\} \prod_{j=1}^{M-1}\{p_{m_j}(z_j)\}$$

$$\times \frac{1}{i2\pi} \int_C d\lambda \exp(-\lambda\, z_M)\, P(z_{M-1},\lambda) \Big/ \prod_{j=1}^{M-1}\{c_{m_j}(z_{M-1},\lambda)\}, \qquad (1)$$

where $U(\,)$ is the unit-step function, C is the contour in the MGF domain λ, $P(z,\lambda)$ is the product function defined in equation (3) of reference 2, and the c-function is

$$c_n(u,\lambda) = \int_{-\infty}^{u} dx \exp(\lambda x)\, p_n(x) \quad \text{for } n = 1:N. \qquad (2)$$

Evaluation of the top line of equation (1) is trivial; however, the bottom line is more involved. An alternative version of the bottom line of equation (1) is

$$\frac{1}{i2\pi} \int_C d\lambda \exp(-\lambda\, z_M) \prod_{n\varepsilon X}\{c_n(z_{M-1},\lambda)\}, \qquad (3)$$

where the notation $n\varepsilon X$ denotes $n = 1{:}N$ except $n \neq m_1, m_2, \ldots, m_{M-1}$.

The logarithm of the integrand in equation (3) is defined as

$$\Lambda(\lambda) = \log\left[\exp(-\lambda\, z_M) \prod_{n\varepsilon X}\{c_n(z_{M-1},\lambda)\}\right]$$

$$= -\lambda\, z_M + \sum_{n\varepsilon X} \log[c_n(z_{M-1},\lambda)]. \qquad (4)$$

The first two derivatives with respect to λ follow as

$$\Lambda'(\lambda) = -z_M + \sum_{n\varepsilon X} \frac{c_n'(z_{M-1},\lambda)}{c_n(z_{M-1},\lambda)} \qquad (5)$$

and

$$\Lambda''(\lambda) = \sum_{n\varepsilon X} \frac{c_n(z_{M-1},\lambda)\, c_n''(z_{M-1},\lambda) - [c_n'(z_{M-1},\lambda)]^2}{[c_n(z_{M-1},\lambda)]^2}. \qquad (6)$$

The saddlepoint (SP) λ_s of $\Lambda(\lambda)$ (and the integrand of equation (3)) is located where

$$\Lambda'(\lambda_s) = 0, \quad \lambda_s = \lambda_s(z_M, z_{M-1}), \tag{7}$$

which is independent of variables $\{z_m\}$, $m < M - 1$, but which does depend on integers $\{m_j\}$, $j = 1{:}M - 1$. The standard SPA to equation (3) is given by

$$SPA0 = \frac{\exp(-\lambda_s\, z_M) \prod_{n \varepsilon X} \{c_n(z_{M-1}, \lambda_s)\}}{[2\pi\, \Lambda''(\lambda_s)]^{1/2}} = \frac{\exp[\Lambda(\lambda_s)]}{[2\pi\, \Lambda''(\lambda_s)]^{1/2}}. \tag{8}$$

The first-order correction to the SPA is

$$SPA1 = SPA0 \left[1 + \frac{1}{8} \frac{\Lambda''''(\lambda_s)}{\Lambda''(\lambda_s)^2} - \frac{5}{24} \frac{\Lambda'''(\lambda_s)^2}{\Lambda''(\lambda_s)^3} \right], \tag{9}$$

and requires two additional derivatives beyond those given in equations (5) and (6).

EXAMPLE 1: EXPONENTIAL RANDOM VARIABLES

From pages A-1 and A-3 of reference 1, there follows the individual PDF

$$p_n(x) = \frac{1}{b_n} \exp\left(-\frac{x}{b_n} \right) \text{ for } x > 0, \quad n = 1 : N \tag{10}$$

and the corresponding c-function

$$c_n(u, \lambda) = \frac{1 - \exp(-u/b_n + u\,\lambda)}{1 - b_n\,\lambda} \text{ for } u > 0; \;\; c_n(u, 1/b_n) = \frac{u}{b_n}. \tag{11}$$

Random variable x_n has mean b_n for $n = 1{:}N$.

At this point, it is useful to define function

$$L(z) = \log\left(\frac{1 - \exp(-z)}{z} \right) \text{ for all } z \neq 0; \;\; L(0) = 0. \tag{12}$$

Then, using equation (11),

$$\log[c_n(u, \lambda)] = \log\left(\frac{u}{b_n} \right) + L\left(\frac{u}{b_n} - u\,\lambda \right), \tag{13}$$

and equation (4) yields

$$\Lambda(\lambda) = -\lambda z_M + \sum_{n\varepsilon X}\left\{\log\left(\frac{z_{M-1}}{b_n}\right) + L\left(\frac{z_{M-1}}{b_n} - z_{M-1}\lambda\right)\right\}. \tag{14}$$

There follows,

$$\Lambda'(\lambda) = -z_M - z_{M-1}\sum_{n\varepsilon X} L'\left(\frac{z_{M-1}}{b_n} - z_{M-1}\lambda\right),$$

$$\Lambda^{(k)}(\lambda) = (-z_{M-1})^k \sum_{n\varepsilon X} L^{(k)}\left(\frac{z_{M-1}}{b_n} - z_{M-1}\lambda\right) \text{ for } k \geq 2. \tag{15}$$

The derivatives required of equation (12) use equations (7), (8), (9), and (15), and are given by

$$\begin{aligned}
L'(z) &= -\frac{1}{z} + \frac{1}{E-1}, \quad E \equiv \exp(z), \\
L''(z) &= \frac{1}{z^2} - \frac{E}{(E-1)^2}, \\
L'''(z) &= -\frac{2}{z^3} + \frac{E(E+1)}{(E-1)^3}, \\
L''''(z) &= \frac{6}{z^4} - \frac{E(E^2+4E+1)}{(E-1)^4}.
\end{aligned} \tag{16}$$

The values of the functions above at $z = 0$ are to be taken as the limits as z approaches zero. All four of these functions are analytic functions of z on the real axis. However, great care must be taken in dealing numerically with the removable singularities at $z = 0$, especially for the higher derivatives. These issues are discussed in appendix A. The high-order poles at $z = i2\pi k$, $k = \pm 1, \pm 2, \ldots$, do not interfere with the search for the SP λ_s of equation (7) on the real λ-axis. That is, $\Lambda'(\lambda)$ in equation (15) encounters purely real arguments for $L'(z)$ in equation (16) when λ is real.

EXAMPLE 2: SQUARED-GAUSSIAN RANDOM VARIABLES

From pages A-2 and A-3 of reference 1, the individual PDF is

$$p_n(x) = \frac{\exp\left(-\dfrac{x}{2\,b_n}\right)}{(2\pi\,x\,b_n)^{1/2}} \quad \text{for } x > 0 \tag{17}$$

and the corresponding c-function is

$$c_n(u,\lambda) = \frac{\operatorname{erf}\left(\sqrt{\dfrac{u}{2\,b_n}}\,\sqrt{1-2\,b_n\,\lambda}\right)}{\sqrt{1-2\,b_n\,\lambda}} \quad \text{for } u>0; \quad c_n\left(u,\frac{1}{2\,b_n}\right) = \left(\frac{2}{\pi}\,\frac{u}{b_n}\right)^{1/2}. \tag{18}$$

Random variable $\boldsymbol{x}_n$ has mean b_n for $n = 1{:}N$.

At this point, the relevant function is defined as

$$L(z) = \log\left(\frac{\operatorname{erf}(\sqrt{z})}{\sqrt{z}}\right) \quad \text{for all } z \neq 0; \quad L(0) = \log(2/\sqrt{\pi}), \tag{19}$$

which is entire. Then, using equation (18),

$$\log[c_n(u,\lambda)] = \frac{1}{2}\log\left(\frac{u}{2\,b_n}\right) + L\left(\frac{u}{2\,b_n} - u\,\lambda\right), \tag{20}$$

and equation (4) yields

$$\Lambda(\lambda) = -\lambda\,z_M + \frac{1}{2}\sum_{n\varepsilon X}\log\left(\frac{z_{M-1}}{2\,b_n}\right) + \sum_{n\varepsilon X} L\left(\frac{z_{M-1}}{2\,b_n} - z_{M-1}\,\lambda\right). \tag{21}$$

There follows

$$\begin{aligned} \Lambda'(\lambda) &= -z_M - z_{M-1}\sum_{n\varepsilon X} L'\left(\frac{z_{M-1}}{2\,b_n} - z_{M-1}\,\lambda\right), \\ \Lambda^{(k)}(\lambda) &= (-z_{M-1})^k \sum_{n\varepsilon X} L^{(k)}\left(\frac{z_{M-1}}{2\,b_n} - z_{M-1}\,\lambda\right) \quad \text{for } k \geq 2. \end{aligned} \tag{22}$$

The derivatives required of equation (19) use equations (7), (8), (9), and (22), and are given by

$$
\begin{aligned}
L'(z) &= -\frac{1}{2z} + G, \quad G \equiv \frac{\exp(-z)}{\sqrt{\pi z}\,\mathrm{erf}(\sqrt{z})}, \\
L''(z) &= \frac{1}{2z^2} - \left(1 + \frac{0.5}{z}\right) G - G^2, \\
L'''(z) &= -\frac{1}{z^3} + \left(1 + \frac{1}{z} + \frac{0.75}{z^2}\right) G + \left(3 + \frac{1.5}{z}\right) G^2 + 2G^3, \\
L''''(z) &= \frac{3}{z^4} - \left(1 + \frac{1.5}{z} + \frac{2.25}{z^2} + \frac{1.875}{z^3}\right) G - \left(7 + \frac{7}{z} + \frac{3.75}{z^2}\right) G^2 - \left(12 + \frac{6}{z}\right) G^3 - 6G^4.
\end{aligned}
\tag{23}
$$

The values of the functions above at $z = 0$ are to be taken as the limits as z approaches zero. All four of these functions are entire functions of z. However, great care must be taken in dealing numerically with the removable singularities at $z = 0$, especially for the higher derivatives. These issues are taken up in appendix B.

EXAMPLE 3: GAUSSIAN RANDOM VARIABLES

From pages A-1 and A-3 of reference 1, the individual PDF is

$$
p_n(x) = \frac{1}{\sqrt{2\pi}\, b_n} \exp\left(-\frac{1}{2}\left(\frac{x - a_n}{b_n}\right)^2\right) \equiv \frac{1}{b_n}\,\phi\left(\frac{x - a_n}{b_n}\right) \tag{24}
$$

and the corresponding c-function is

$$
c_n(u,\lambda) = \exp\left(a_n\,\lambda + \frac{1}{2} b_n^2\,\lambda^2\right) \Phi\left(\frac{u - a_n}{b_n} - b_n\,\lambda\right), \tag{25}
$$

where Φ is the cumulative distribution function of a normalized Gaussian RV. That is,

$$
\Phi(t) = \int_{-\infty}^{t} du\; \phi(u) = \int_{-\infty}^{t} du\; (2\pi)^{-1/2} \exp(-u^2/2). \tag{26}
$$

Random variable $\mathbf{x}_n$ has mean a_n and standard deviation b_n for $n = 1{:}N$ (see equation (24)).

At this point, the relevant functions are defined as

$$L(\lambda) = a\lambda + \frac{1}{2}b^2\lambda^2 + \log\Phi(A(\lambda)), \quad A(\lambda) = \frac{u-a}{b} - b\,\lambda, \quad A'(\lambda) = -b, \tag{27}$$

which are entire. Then, the first derivative is given by

$$L'(\lambda) = a + b^2\lambda - b\,r(\lambda), \tag{28}$$

where

$$r(\lambda) \equiv \frac{\phi(A(\lambda))}{\Phi(A(\lambda))}, \quad r'(\lambda) = b\,r(\lambda)\,[A(\lambda) + r(\lambda)]. \tag{29}$$

The remaining higher derivatives follow upon reapplication of equation (29) to equation (28), namely,

$$\begin{aligned} L''(\lambda) &= b^2 - b^2\,r(\lambda)\,[A(\lambda) + r(\lambda)], \\ L'''(\lambda) &= b^3\,r(\lambda)\,[1 - A^2(\lambda) - 3\,A(\lambda)r(\lambda) - 2\,r^2(\lambda)], \\ L''''(\lambda) &= b^4\,r(\lambda)\,[3\,A(\lambda) - A^3(\lambda) + (4 - 7A^2(\lambda))\,r(\lambda) - 12\,A(\lambda)\,r^2(\lambda) - 6\,r^3(\lambda)]. \end{aligned} \tag{30}$$

The quantities $A(\lambda)$ and $r(\lambda)$ are given in equations (27) and (29), respectively.

The corresponding derivatives of equation (4) are now available as

$$\begin{aligned} \Lambda(\lambda) &= -\lambda\,z_M + \sum_{n\varepsilon X}[a_n\,\lambda + \tfrac{1}{2}b_n^2\,\lambda^2 + \log\Phi(A_n)], \quad A_n = A_n(\lambda) = \frac{z_{M-1} - a_n}{b_n} - b_n\,\lambda, \\ \Lambda'(\lambda) &= -z_M + \sum_{n\varepsilon X}[a_n + b_n^2\,\lambda - b_n\,r_n], \quad r_n = r_n(\lambda) = \frac{\phi(A_n(\lambda))}{\Phi(A_n(\lambda))}, \\ \Lambda''(\lambda) &= \sum_{n\varepsilon X} b_n^2 - \sum_{n\varepsilon X} b_n^2\,r_n\,[A_n + r_n], \\ \Lambda'''(\lambda) &= \sum_{n\varepsilon X} b_n^3\,r_n\,[1 - A_n^2 - 3\,A_n\,r_n - 2\,r_n^2], \\ \Lambda''''(\lambda) &= \sum_{n\varepsilon X} b_n^4\,r_n\,[3\,A_n - A_n^3 + (4 - 7A_n^2)\,r_n - 12\,A_n\,r_n^2 - 6\,r_n^3]. \end{aligned} \tag{31}$$

EXAMPLE 4: CHI-SQUARED RANDOM VARIABLES OF $2k+2$ DEGREES OF FREEDOM

From pages A-2 and A-3 of reference 1, the individual PDF is

$$p_n(x) = \frac{x^k \exp(-x/b_n)}{b_n^{k+1}\, k!} \quad \text{for } x > 0,\ \ k \text{ integer} \geq 0 \tag{32}$$

and the corresponding c-function is

$$c_n(u,\lambda) = \frac{1 - \exp(-u/b_n + u\,\lambda) \sum_{j=0}^{k} \frac{1}{j!} (u/b_n - u\,\lambda)^j}{(1 - b_n\,\lambda)^{k+1}} \quad \text{for } u > 0;$$

$$c_n(u, 1/b_n) = \frac{u^{k+1}}{b_n^{k+1}\,(k+1)!}. \tag{33}$$

Random variable x_n has mean $(k+1)\, b_n$ for $n = 1{:}N$.

Define function

$$L(z) = \log\left(\frac{1 - \exp(-z) \sum_{j=0}^{k} z^j/j!}{z^{k+1}} \right) \quad \text{for } z \neq 0; \quad L(0) = -\log((k+1)!). \tag{34}$$

Then, using equation (33),

$$\log[c_n(u,\lambda)] = (k+1)\log\left(\frac{u}{b_n}\right) + L\left(\frac{u}{b_n} - u\,\lambda\right) \tag{35}$$

and equation (4) yields

$$\Lambda(\lambda) = -\lambda\, z_M + (k+1) \sum_{n \in X} \log\left(\frac{z_{M+1}}{b_n}\right) + \sum_{n \in X} L\left(\frac{z_{M+1}}{b_n} - z_{M+1}\,\lambda\right). \tag{36}$$

There follows

$$\Lambda'(\lambda) = -z_M - z_{M-1} \sum_{n\varepsilon X} L'\left(\frac{z_{M-1}}{b_n} - z_{M-1}\,\lambda\right),$$

$$\Lambda^{(p)}(\lambda) = (-z_{M-1})^p \sum_{n\varepsilon X} L^{(p)}\left(\frac{z_{M-1}}{b_n} - z_{M-1}\,\lambda\right) \text{ for } p \geq 2. \tag{37}$$

At this point, it is necessary to consider specific cases for integer k in equations (32) through (34). The case of $k = 0$ was discussed in equation (10) and the text that followed. The first case of interest here is $k = 1$, namely, $2k+2 = 4$ degrees of freedom (DOF). The function of interest and its first four derivatives are

$$\begin{aligned}
L(z) &= \log\left(\frac{1-\exp(-z)\,(1+z)}{z^2}\right) \text{ for } z \neq 0, \quad L(0) = -\log(2), \\
L'(z) &= -\frac{2}{z} + \frac{z}{E-1-z}, \quad E \equiv \exp(z), \\
L''(z) &= \frac{2}{z^2} + \frac{E\,(1-z)-1}{(E-1-z)^2}, \\
L'''(z) &= -\frac{4}{z^3} + \frac{E^2\,(-2+z) + E\,(4-z+z^2) - 2}{(E-1-z)^3}, \\
L''''(z) &= \frac{12}{z^4} + \frac{E^3\,(3-z) + 4\,E^2\,(-3+2z-z^2) + E\,(15-7z+z^2-z^3) - 6}{(E-1-z)^4}.
\end{aligned} \tag{38}$$

The second case is $k = 2$, which corresponds to six DOF:

$$\begin{aligned}
L(z) &= \log\left(\frac{1-\exp(-z)\,(1+z+z^2/2)}{z^3}\right) \text{ for } z \neq 0, \quad L(0) = -\log(6), \\
L'(z) &= -\frac{3}{z} + \frac{z^2}{2\,D}, \quad E \equiv \exp(z), \quad D \equiv E - (1+z+z^2/2), \\
L''(z) &= \frac{3}{z^2} + z\frac{E\,(2-z)-2-z}{2\,D^2}, \\
L'''(z) &= -\frac{6}{z^3} + \frac{E^2\,(4-8z+2z^2) + E\,(-8+8z+8z^2-2z^3+z^4) + 4 - 6z^2 - 2z^3}{4\,D^3}, \\
L''''(z) &= \frac{18}{z^4} + \frac{1}{8\,D^4}\Big(4\,E^3\,(-6+6z-z^2) + 8\,E^2\,(9-8z^2+4z^3-z^4) \\
&\quad + E\,(-72-72z+68z^2+16z^3-14z^4+2z^5-z^6) + 6(4+8z-4z^3-z^4)\Big)
\end{aligned} \tag{39}$$

The third case is $k = 3$, which corresponds to 8 DOF:

$$L(z) = \log\left(\frac{1 - \exp(-z)\,(1 + z + z^2/2 + z^3/6)}{z^4}\right) \quad \text{for } z \neq 0, \quad L(0) = -\log(24),$$

$$L'(z) = -\frac{4}{z} + \frac{z^3}{6D}, \quad E \equiv \exp(z), \quad D \equiv E - (1 + z + z^2/2 + z^3/6),$$

$$L''(z) = \frac{4}{z^2} + \frac{z^2}{12D^2}\left(E(6 - 2z) - (6 + 4z + z^2)\right),$$

$$L'''(z) = -\frac{8}{z^3} + \frac{z}{36D^3}\left(6E^2(6 - 6z + z^2) + E(-72 + 24z^2 + 12z^3 - 3z^4 + z^5)\right. \tag{40}$$
$$\left. + 36 + 36z + 6z^2 - 12z^3 - 6z^4 - z^5\right),$$

$$L''''(z) = \frac{1}{216D^4}\left(36E^3(6 - 18z + 9z^2 - z^3) + 24E^2(-27 + 54z + 27z^2 - 15z^4 + 6z^5 - z^6)\right.$$
$$+ E(648 - 648z - 1620z^2 - 1188z^3 + 126z^4 + 162z^5 + 3z^6 - 21z^7 + 3z^8 - z^9)$$
$$\left. - 3(72 - 216z^2 - 336z^3 - 186z^4 - 24z^5 + 20z^6 + 8z^7 + z^8)\right) + \frac{24}{z^4}.$$

The method of handling the high-order removable singularities of equations (38), (39), and (40) at $z = 0$ is discussed in appendix C.

SEARCHING FOR THE SADDLEPOINT

The SP location of the integrand of equation (3) is on the real λ axis. Furthermore, the integrand has a single minimum on the real axis, in its region of analyticity, which is the SP location λ_s. This will be demonstrated for arbitrary PDFs $\{p_n(x)\}$ and arbitrary M. It is presumed that $z_M < (N+1-M)\, z_{M-1}$ so that the joint PDF of interest is nonzero.

Equation (6) is the starting point for the second derivative of the logarithm of the integrand of equation (3). In general,

$$c(u,\lambda) = \int_{-\infty}^{u} dx\; p(x)\exp(\lambda x),$$

$$c'(u,\lambda) = \int_{-\infty}^{u} dx\; p(x)\exp(\lambda x)\, x, \tag{41}$$

$$c''(u,\lambda) = \int_{-\infty}^{u} dx\; p(x)\exp(\lambda x)\, x^2.$$

Then, the numerator of equation (6) can be expressed as

$$\left[\int_{-\infty}^{u} dx\; p(x)\exp(\lambda x)\right]\left[\int_{-\infty}^{u} dx\; p(x)\exp(\lambda x)\, x^2\right]-\left[\int_{-\infty}^{u} dx\; p(x)\exp(\lambda x)\, x\right]^2 . \tag{42}$$

But, Schwartz's inequality states that

$$\left[\int dx\, f(x)\, g(x)\right]^2 \le \left[\int dx\, f^2(x)\right]\left[\int dx\, g^2(x)\right], \tag{43}$$

with equality if and only if $f(x)$ and $g(x)$ are proportional for all x. Upon identification of

$$f(x)=\sqrt{p(x)}\,\exp(\lambda x/2) \quad \text{and} \quad g(x)=\sqrt{p(x)}\,\exp(\lambda x/2)\, x, \tag{44}$$

it follows that equation (42) is positive for all λ. Therefore, every numerator of equation (6) is positive, meaning that $\Lambda''(\lambda) > 0$ for all real λ. That is, $\Lambda(\lambda)$ is bowl-like, for real λ, with a single minimum in its original region of definition (convergence of its integral). Therefore, the integrand of equation (3), namely $\exp[\Lambda(\lambda)]$, has the same property.

The above conclusion holds for all values of z_M and z_{M-1}. However, the actual location of the minimum of $\Lambda(\lambda)$ depends on both of these variables, as can be seen by setting equation (5) to zero (see equation (7)). Variables $\{z_m\}$ for $m < M-1$ have no effect on SP location λ_s.

RELATIVE ACCURACY OF THE ESTIMATE OF THE SADDLEPOINT APPROXIMATION

The MGF corresponding to PDF $p(u)$ is given by

$$\mu(\lambda) = \int du \exp(\lambda u)\ p(u) \tag{45}$$

and the inversion formula is

$$p(u) = \frac{1}{i2\pi} \int_C d\lambda \exp(-\lambda u)\ \mu(\lambda), \tag{46}$$

where Bromwich contour C is a vertical line from $-i\infty$ to $+i\infty$ in the region of definition (analyticity) of MGF $\mu(\lambda)$, as obtained from convergence of the integral in equation (45). The logarithm of the integrand of equation (46) is defined as

$$\Lambda(\lambda,u) = -\lambda u + \chi(\lambda), \quad \text{where} \quad \chi(\lambda) \equiv \log \mu(\lambda). \tag{47}$$

The SPA to exact PDF $p(u)$ in equation (46) is then

$$p_s(u) = \frac{\exp(-\lambda_s u)\ \mu(\lambda_s)}{[2\pi\, \chi''(\lambda_s)]^{1/2}}, \tag{48}$$

where SP location $\lambda_s = \lambda_s(u)$ satisfies the SP equation, $\chi'(\lambda_s) = u$.

Now, in the neighborhood of the actual SP,

$$\begin{aligned} &\chi'(\lambda) \cong \chi'(\lambda_s) + \chi''(\lambda_s)(\lambda - \lambda_s) \quad \text{for } \lambda \text{ near } \lambda_s, \\ &\chi'(\lambda) - u \cong \chi''(\lambda_s)(\lambda - \lambda_s). \end{aligned} \tag{49}$$

Let a *relative* error tolerance R_a be imposed on the accuracy of the SP location, namely,

$$\left| \frac{\chi'(\lambda) - u}{u} \right| < R_a; \quad u \neq 0. \tag{50}$$

Then, the solution λ to the SP equation will satisfy

$$\left| \frac{\chi''(\lambda_s)(\lambda - \lambda_s)}{u} \right| < R_a, \quad |\lambda - \lambda_s| < \frac{R_a\, |u|}{\chi''(\lambda_s)}. \tag{51}$$

Therefore, let solution λ_c (the axis crossing actually *used* for the Bromwich contour C) be

$$\lambda_c = \lambda_s + \varepsilon, \quad \text{where} \quad |\varepsilon| < \frac{R_a\,|u|}{\chi''(\lambda_s)}. \tag{52}$$

Adopt the *estimate* of the SPA as

$$p_c(u) \equiv \frac{\exp(-\lambda_c\,u)\;\mu(\lambda_c)}{[2\pi\,\chi''(\lambda_c)]^{1/2}}, \tag{53}$$

in analogy with equation (48). Then,

$$\exp(-\lambda_c\,u) = \exp(-(\lambda_s + \varepsilon)\,u) = \exp(-\lambda_s\,u)\,\exp(-\varepsilon\,u) \cong \exp(-\lambda_s\,u)\,(1 - \varepsilon\,u),$$

$$\begin{aligned} \mu(\lambda_c) &= \mu(\lambda_s + \varepsilon) \cong \mu(\lambda_s) + \varepsilon\,\mu'(\lambda_s) = \mu(\lambda_s)\left[1 + \varepsilon\frac{\mu'(\lambda_s)}{\mu(\lambda_s)}\right] \\ &= \mu(\lambda_s)\,[1 + \varepsilon\,\chi'(\lambda_s)] = \mu(\lambda_s)\,(1 + \varepsilon\,u), \end{aligned} \tag{54}$$

$$\exp(-\lambda_c\,u)\;\mu(\lambda_c) \cong \exp(-\lambda_s\,u)\;\mu(\lambda_s)\,(1 - \varepsilon^2\,u^2),$$

which is *quadratic* in ε. This is consistent with λ_s being a SP of the integrand of equation (46). Also,

$$\begin{aligned} &\chi''(\lambda_c) = \chi''(\lambda_s + \varepsilon) \cong \chi''(\lambda_s) + \varepsilon\,\chi'''(\lambda_s), \\ &[\chi''(\lambda_c)]^{1/2} \cong [\chi''(\lambda_s)]^{1/2}\left[1 + \frac{\varepsilon}{2}\frac{\chi'''(\lambda_s)}{\chi''(\lambda_s)}\right]. \end{aligned} \tag{55}$$

Combining these results with equations (53) through (55), there follows

$$p_c(u) \cong p_s(u)\left[1 - \frac{\varepsilon}{2}\frac{\chi'''(\lambda_s)}{\chi''(\lambda_s)}\right] \quad \text{to order } \varepsilon. \tag{56}$$

Therefore, using equation (52), the relative error in the estimated SPA $p_c(u)$ is

$$R_1 \equiv \left|\frac{\varepsilon}{2}\frac{\chi'''(\lambda_s)}{\chi''(\lambda_s)}\right| < \frac{R_a\,|u|}{2}\,\frac{|\chi'''(\lambda_s)|}{\chi''(\lambda_s)^2}. \tag{57}$$

Recall that SP location $\lambda_s = \lambda_s(u)$ is a function of u. Therefore, R_1 can have a complicated dependence on field point u. The third derivative of χ at λ_s only needs to be evaluated *once*, after the SP location has been determined. Ratio R_1/R_a is called the error factor.

The modified correction term (CT) to the SPA is

$$CT = \frac{1}{8}\frac{\chi''''(\lambda_c)}{\chi''(\lambda_c)^2} - \frac{5}{24}\frac{\chi'''(\lambda_c)^2}{\chi''(\lambda_c)^3}; \quad \widetilde{p}_c(u) = p_c(u)\,[1 + CT]. \tag{58}$$

Upon expanding these functions according to

$$\begin{gathered} \chi''(\lambda_c) = \chi''(\lambda_s + \varepsilon) \cong \chi''(\lambda_s) + \varepsilon\,\chi'''(\lambda_s) \equiv \chi_2 + \varepsilon\,\chi_3, \\ \chi'''(\lambda_c) \cong \chi_3 + \varepsilon\,\chi_4, \quad \chi''''(\lambda_c) \cong \chi_4 + \varepsilon\,\chi_5, \end{gathered} \tag{59}$$

there follows

$$CT \cong \frac{1}{8}\frac{\chi_4}{\chi_2^2}\left[1 + \varepsilon\left(\frac{\chi_5}{\chi_4} - 2\frac{\chi_3}{\chi_2}\right)\right] - \frac{5}{24}\frac{\chi_3^2}{\chi_2^3}\left[1 + \varepsilon\left(2\frac{\chi_4}{\chi_3} - 3\frac{\chi_3}{\chi_2}\right)\right]. \tag{60}$$

Thus, the perturbation in CT is also linearly proportional to ε.

An alternative method of determining SP location λ_s is to solve for the minimum of the integrand of equation (46) on the real λ-axis. Equivalently, the minimum of equation (47), which is the logarithm of the integrand, will suffice. For the latter function, there follows

$$\Lambda(\lambda,u) \cong \Lambda(\lambda_s,u) + \Lambda'(\lambda_s,u)\,(\lambda - \lambda_s) + \tfrac{1}{2}\Lambda''(\lambda_s,u)\,(\lambda - \lambda_s)^2 \tag{61}$$

for λ near λ_s. But, use of the relations

$$\Lambda'(\lambda_s,u) = -u + \chi'(\lambda_s) = 0, \quad \Lambda''(\lambda_s,u) = \chi''(\lambda_s), \tag{62}$$

leads to the result

$$\Lambda(\lambda,u) \cong \Lambda(\lambda_s,u) + \tfrac{1}{2}\chi''(\lambda_s)\,(\lambda - \lambda_s)^2. \tag{63}$$

Therefore, when a relative error tolerance is imposed on the search for the minimum, namely,

$$\left|\frac{\Lambda(\lambda,u) - \Lambda(\lambda_s,u)}{\Lambda(\lambda_s,u)}\right| < R_b, \tag{64}$$

the requirement yields

$$|\lambda - \lambda_s| < \left(\frac{2 R_b |\Lambda(\lambda_s, u)|}{\chi''(\lambda_s)} \right)^{1/2} . \tag{65}$$

Now, let solution λ_c (the axis crossing actually *used* for the Bromwich contour C) be

$$\lambda_c = \lambda_s + \varepsilon, \quad \text{where} \quad |\varepsilon| < \left(\frac{2 R_b |\Lambda(\lambda_s, u)|}{\chi''(\lambda_s)} \right)^{1/2} . \tag{66}$$

Then, the analysis in equations (53) through (56) is unaltered, except that now

$$R_2 \equiv \left| \frac{\varepsilon}{2} \frac{\chi'''(\lambda_s)}{\chi''(\lambda_s)} \right| < \left(\frac{R_b}{2} |\Lambda(\lambda_s, u)| \frac{\chi'''(\lambda_s)^2}{\chi''(\lambda_s)^3} \right)^{1/2} \tag{67}$$

is the relative error of the estimated SPA $p_c(u)$. A comparison with equation (57) reveals that relative error R_1 depends linearly on tolerance R_a, whereas relative error R_2 varies as the square root of tolerance R_b. This degradation in the quality of R_2 is due to the fact that searching for a minimum of a function is fundamentally less accurate than searching for a zero of its derivative. The minimum search is at the bottom of a parabolic bowl, where much larger changes in the abscissa are required than at the zero crossing of a nearly linear function.

The bounds in equations (57) and (67) depend on high-order derivatives of $\chi(\lambda) = \log \mu(\lambda)$ evaluated at the SP location λ_s. All of these quantities depend upon the value of the field point u as well as the particular PDF $p(u)$ under investigation. No general statements about accuracy are apparent; accordingly, a couple of examples will be investigated quantitatively.

EXAMPLE 1: EXPONENTIAL RANDOM VARIABLE

$$\begin{aligned}
&p(u) = \exp(-u) \quad \text{for } u > 0, \\
&\mu(\lambda) = \frac{1}{1-\lambda} \quad \text{for } \mathrm{Re}(\lambda) < 1, \\
&\Lambda(\lambda, u) = -\lambda u - \log(1-\lambda), \\
&\chi(\lambda) = -\log(1-\lambda), \;\; \chi'(\lambda) = \frac{1}{1-\lambda}, \;\; \chi^{(n)}(\lambda) = \frac{(n-1)!}{(1-\lambda)^n} \quad \text{for } n \ge 1.
\end{aligned} \tag{68}$$

All these relations require $\mathrm{Re}(\lambda) < 1$. For $u > 0$, the SP equation yields

$$\chi'(\lambda_s) = u, \ \frac{1}{1-\lambda_s} = u, \ \lambda_s = 1 - \frac{1}{u},$$
$$\chi''(\lambda_s) = \frac{1}{(1-\lambda_s)^2} = u^2, \ \chi'''(\lambda_s) = 2u^3, \ \chi''''(\lambda_s) = 6u^4, \tag{69}$$
$$\frac{\chi'''(\lambda_s)}{\chi''(\lambda_s)^2} = \frac{2}{u}.$$

Then, the relative error in equation (57) is upper-bounded by

$$R_1 < R_a \frac{u}{2}\frac{2}{u} = R_a. \tag{70}$$

That is, the relative error in estimated SPA $p_c(u)$ is less than that in determining SP location λ_s itself, by means of the SP equation $\chi'(\lambda_s) = u$.

By use of equations (68) and (69), there follows

$$\chi_n \equiv \chi^{(n)}(\lambda_s) = \frac{(n-1)!}{(1-\lambda_s)^n} = (n-1)!\, u^n,$$
$$\frac{\chi_5}{\chi_4} - 2\frac{\chi_3}{\chi_2} = \frac{24u^5}{6u^4} - 2\frac{2u^3}{u^2} = 0, \tag{71}$$
$$2\frac{\chi_4}{\chi_3} - 3\frac{\chi_3}{\chi_2} = 2\frac{6u^4}{2u^3} - 3\frac{2u^3}{u^2} = 0.$$

Thus, the correction term (60) is unaffected; that is,

$$\frac{1}{8}\frac{\chi''''(\lambda)}{\chi''(\lambda)^2} - \frac{5}{24}\frac{\chi'''(\lambda)^2}{\chi''(\lambda)^3} = \frac{1}{8}\frac{6/(1-\lambda)^4}{1/(1-\lambda)^4} - \frac{5}{24}\frac{4/(1-\lambda)^6}{1/(1-\lambda)^6} = \frac{3}{4} - \frac{5}{6} = -\frac{1}{12} \tag{72}$$

for all λ. The correction term is independent of λ or λ_s or λ_c. The SPA is

$$p_s(u) = \exp(-u)\frac{e}{\sqrt{2\pi}} = 1.0844 \exp(-u) \quad \text{for } u > 0. \tag{73}$$

For the alternative approach, where $\Lambda(\lambda, u)$ is minimized, the relevant relations are

$$\Lambda(\lambda_s, u) = -\lambda_s u - \log(1 - \lambda_s) = 1 - u + \log(u) \tag{74}$$

and

$$\frac{\chi'''(\lambda_s)^2}{\chi''(\lambda_s)^3} = \frac{(2u^3)^2}{(u^2)^3} = 4, \tag{75}$$

giving

$$R_2 < (2\,R_b\,|1-u+\log(u)|)^{1/2}. \tag{76}$$

One bad effect in this approach is the dependence on $R_b^{1/2}$ instead of R_b; this is due to the quadratic behavior of $\Lambda(\lambda,u)$ near SP location λ_s. The second bad effect is the particular dependence on u, namely,

$$R_2 \sim \left\{\begin{matrix} (2\,R_b\,|\log(u)|)^{1/2} & \text{as}\ u \to 0+ \\ (2\,R_b\,u)^{1/2} & \text{as}\ u \to +\infty \end{matrix}\right\}. \tag{77}$$

Although the singularity near $u = 0$ is not likely to cause problems because of its weak behavior, the singularity for large u could be problematical, especially if u is the order of $1/R_b >> 1$. Even if $u = 0.01/R_b$, the relative error is $(2/100)^{1/2} = 0.14$, which is a noticeable relative error.

EXAMPLE 2: GAMMA RANDOM VARIABLE

$$\begin{aligned}
&p(u) = \frac{u^k \exp(-u)}{\Gamma(k+1)} \quad \text{for}\ u > 0,\ \ k > -1, \\
&\mu(\lambda) = \frac{1}{(1-\lambda)^{k+1}} \quad \text{for}\ \mathrm{Re}(\lambda) < 1, \\
&\chi(\lambda) = -(k+1)\log(1-\lambda), \\
&\Lambda(\lambda,u) = -\lambda\,u - (k+1)\log(1-\lambda), \\
&\chi'(\lambda) = \frac{k+1}{1-\lambda}, \quad \chi^{(n)}(\lambda) = (k+1)\,\frac{(n-1)!}{(1-\lambda)^n} \quad \text{for}\ n \geq 1.
\end{aligned} \tag{78}$$

For $u > 0$,

$$\begin{aligned}
&\chi'(\lambda_s) = u, \quad \frac{k+1}{1-\lambda_s} = u, \quad 1-\lambda_s = \frac{k+1}{u}, \quad \lambda_s = 1 - \frac{k+1}{u}, \\
&\chi''(\lambda_s) = \frac{u^2}{k+1}, \quad \chi'''(\lambda_s) = \frac{2u^3}{(k+1)^2}, \quad \chi''''(\lambda_s) = \frac{6u^4}{(k+1)^3}, \\
&\frac{\chi'''(\lambda_s)}{\chi''(\lambda_s)^2} = \frac{2}{u}, \quad \text{independent of}\ k.
\end{aligned} \tag{79}$$

Then, the relative error in equation (57) is upper-bounded according to

$$R_1 < \frac{R_a u}{2} \frac{2}{u} = R_a, \tag{80}$$

independent of k.

At the same time, by the use of equations (78) and (79),

$$\chi_n \equiv \chi^{(n)}(\lambda_s) = \frac{(n-1)!\, u^n}{(k+1)^{n-1}}, \tag{81}$$

giving rise to

$$\frac{\chi_5}{\chi_4} - 2\frac{\chi_3}{\chi_2} = 0, \quad 2\frac{\chi_4}{\chi_3} - 3\frac{\chi_3}{\chi_2} = 0. \tag{82}$$

Thus, the correction term is unaffected; that is,

$$\frac{1}{8}\frac{\chi''''(\lambda)}{\chi''(\lambda)^2} - \frac{5}{24}\frac{\chi'''(\lambda)^2}{\chi''(\lambda)^3} = -\frac{1}{12}\frac{1}{k+1} \tag{83}$$

for all λ. The correction term is independent of λ or λ_s or λ_c.

The SPA is

$$p_s(u) = \frac{u^k \exp(-u)}{\Gamma(k+1)} \; \frac{\exp(k+1)\,\Gamma(k+1)}{\sqrt{2\pi}\,(k+1)^{k+1/2}} \equiv p(u)\, g(k) \quad \text{for } u > 0. \tag{84}$$

In particular,

$$\begin{gathered} g(0) = 1.0844, \quad g(1) = 1.0422, \quad g(2) = 1.0281, \quad g(10) = 1.0076, \quad g(\infty) = 1, \\ g(k) \sim 1 + \frac{1}{12k} - \frac{23}{288k^2} + \frac{3821}{51840k^3} \quad \text{as } k \to +\infty. \end{gathered} \tag{85}$$

For the alternative approach, where $\Lambda(\lambda,u)$ is minimized,

$$\Lambda(\lambda_s,u) = -\lambda_s\, u - (k+1)\log(1-\lambda_s) = -u + k + 1 - (k+1)\log\left(\frac{k+1}{u}\right) \tag{86}$$

and

$$\frac{\chi'''(\lambda_s)^2}{\chi''(\lambda_s)^3} = \frac{4}{k+1}, \tag{87}$$

which yields

$$R_2 < \left(2\,R_b \left| 1 - \frac{u}{k+1} + \log\left(\frac{u}{k+1} \right) \right| \right)^{1/2} . \tag{88}$$

In particular,

$$R_2 \sim \left(2\,R_b \,\frac{u}{k+1} \right)^{1/2} \quad \text{as } u \to +\infty. \tag{89}$$

That is, larger k values of the Gamma PDF ameliorate the accuracy problems present for large values of u. The $R_b^{1/2}$ dependence is still present, however.

It has been observed numerically that the SPA is very accurate for evaluating the joint PDF of interest here; in fact, the correction term is virtually 1 in most cases. The reason for this is that the integral in equation (3) pertains to the last RV, which is the sum r of the remainder. For a large number N of RVs, and for M not too large, the Central Limit Theorem says that r (the sum of $N + 1 - M$ RVs) tends to a Gaussian RV. But the SPA itself is *exact* for Gaussian RVs. Therefore, since r is nearly Gaussian, it is expected that the SPA should be quite accurate; this is indeed the case.

PROGRAMS FOR THE SADDLEPOINT APPROXIMATIONS

Six MATLAB programs for the SPAs in equations (8) and (9) are listed below. They are keyed to the four examples listed in the Saddlepoint Approximation section and to tables 1 through 6.

Example 1:	spa_M_log_1 (table 1)
Example 2:	spa_M_log_2 (table 2)
Example 3:	spa_M_log_3 (table 3)
Example 4 (chi-squared, four DOF):	spa_M_log_4_cs4 (table 4)
Example 4 (chi-squared, six DOF):	spa_M_log_4_cs6 (table 5)
Example 4 (chi-squared, eight DOF):	spa_M_log_4_cs8 (table 6).

The log reference in the file names indicates that the logarithm of the SPA is actually calculated, rather than the SPA itself. This avoids underflow and overflow when extreme tail probabilities are of interest. The abbreviation csp stands for chi-squared of p DOF, p integer.

The logarithm of the standard SPA is given by variable spa0_log, while the corresponding first-order correction term is given by spa1_log. The error_factor was defined just below equation (57). Fourth-order derivatives of the various L functions are required to evaluate these expressions. These quantities are thoroughly considered in appendixes A, B, and C.

Table 1. Program for Example 1

```
function spa_M_log_1                    % TR 11,469, general M
global zM zM1 bd                        % Exponential RVs, CS 2 DOF

N=1024;                                 % Number of RVs
n=[1:N]';
b=exp(-.3*(n-.7*N).^2)+.1;              % N means

m=[10;11;12;13]; z=[1.3;.8;.6;.4;100];

M=length(z);
zd=diff(z);
if(any(zd(1:M-2)>0)) pdf=0, keyboard, end
zM=z(M);
zM1=z(M-1);
if(zM>(N+1-M)*zM1) pdf=0, keyboard, end

nd=setdiff(n,m);     % n ~= m(1),...,m(M-1)
bd=b(nd);

options=optimset('tolx',1e-15);
[Ls,fv,ex,ou]=fzero(@Lam1,0,options);
saddlepoint=Ls

ap=z(1:M-1)./b(m);                      % Leading
pm_log=-sum(ap)-sum(log(b(m)));         % PDFs

as=zM1./bd-zM1*Ls;
Lam=-zM*Ls+sum(log(zM1./bd))+sum(Lcs2z(as));
Lam2=zM1^2*sum(Lcs2z2(as));
Lam3=-zM1^3*sum(Lcs2z3(as));
Lam4=zM1^4*sum(Lcs2z4(as));
spa0_log=pm_log+Lam-.5*log(2*pi*Lam2)
ct=.125*Lam4/Lam2^2-5/24*Lam3^2/Lam2^3;
spa1_log=spa0_log+log(1+ct)
error_factor=.5*zM*abs(Lam3)/Lam2^2
keyboard

function w = Lam1(L)     % Lambda'(L)
global zM zM1 bd
w=-zM-zM1*sum(Lcs2z1(zM1./bd-zM1*L));
```

Table 2. Program for Example 2

```
function spa_M_log_2                      % TR 11,469, general M
global zM zM1 bd                          % Gaussian-squared RVs

N=128;                                    % Number of RVs
n=[1:N]';
b=exp(-.3*(n-.7*N).^2)+.1;                % N scale factors, page A-3

m=[10;11;12;13]; z=[1.3;.8;.6;.4;10];

M=length(z);
zd=diff(z);
if(any(zd(1:M-2)>0)) pdf=0, keyboard, end
zM=z(M);
zM1=z(M-1);
if(zM>(N+1-M)*zM1) pdf=0, keyboard, end

nd=setdiff(n,m);     % n ~= m(1),...,m(M-1)
bd=b(nd);

options=optimset('tolx',1e-10);
[Ls,fv,ex,ou]=fzero(@Lam1,0,options);
saddlepoint=Ls

ap=.5*z(1:M-1)./b(m);                     % Leading PDFs
pm_log=-sum(ap)-.5*sum(log(2*pi*z(1:M-1).*b(m)));

as=.5*zM1./bd-zM1*Ls;
Lam=-zM*Ls+.5*sum(log(.5*zM1./bd))+sum(Lerfz(as));
Lam2=zM1^2*sum(Lerfz2(as));
Lam3=-zM1^3*sum(Lerfz3(as));
Lam4=zM1^4*sum(Lerfz4(as));
spa0_log=pm_log+Lam-.5*log(2*pi*Lam2)
ct=.125*Lam4/Lam2^2-5/24*Lam3^2/Lam2^3;
spa1_log=spa0_log+log(1+ct)
error_factor=.5*zM*abs(Lam3)/Lam2^2
keyboard


function w = Lam1(L)     % Lambda'(L)
global zM zM1 bd
w=-zM-zM1*sum(Lerfz1(.5*zM1./bd-zM1*L));
```

Table 3. Program for Example 3

```
function spa_M_log_3                  % TR 11,469, general M
global zM zM1 ad bd                   % Gaussian RVs

N=128;                                % Number of RVs
n=[1:N]';
a=1+n/N;                              % N shift factors, page A-3
b=exp(-.3*(n-.7*N).^2)+.1;            % N scale factors, page A-3

m=[117;126;89]; z=[2.5;2.1;2.05;185];

M=length(z);
zd=diff(z);
if(any(zd(1:M-2)>0)) pdf=0, keyboard, end
zM=z(M);
zM1=z(M-1);
if(zM>(N+1-M)*zM1) pdf=0, keyboard, end

nd=setdiff(n,m);      % n ~= m(1),...,m(M-1)
ad=a(nd); bd=b(nd);

options=optimset('tolx',1e-10);
[Ls,fv,ex,ou]=fzero(@Lam1,0,options);
saddlepoint=Ls

ap=(z(1:M-1)-a(m))./b(m);             % Leading PDFs
pm_log=-.5*sum(ap.^2)-sum(log(sqrt(2*pi)*b(m)));

As=(zM1-ad)./bd-bd*Ls; As2=As.^2;
phis=.5*erfcore(-As*sqrt(.5),1);
rs=sqrt(.5/pi)*exp(-.5*As2)./phis;
bd2=bd.^2; bd2r=bd2.*rs;
Lam=-zM*Ls+sum(Ls*ad+.5*Ls^2*bd2+log(phis));
Lam2=sum(bd2-bd2r.*(As+rs));
Lam3=sum(bd.*bd2r.*(1-As2-rs.*(3*As+2*rs)));
Lam4=sum(bd2.*bd2r.*(As.*(3-As2)+rs.*(4-7*As2-rs.*(12*As+6*rs))));
spa0_log=pm_log+Lam-.5*log(2*pi*Lam2)
ct=.125*Lam4/Lam2^2-5/24*Lam3^2/Lam2^3;
spa1_log=spa0_log+log(1+ct)
error_factor=.5*zM*abs(Lam3)/Lam2^2
keyboard


function w = Lam1(L)     % Lambda'(L)
global zM zM1 ad bd
A=(zM1-ad)./bd-bd*L;
phiA=.5*erfcore(-A*sqrt(.5),1);
r=sqrt(.5/pi)*exp(-.5*A.^2)./phiA;
w=-zM+sum(ad+bd.^2*L-bd.*r);
```

Table 4. Program for Example 4, Four DOF

```
function spa_M_log_4_cs4                % TR 11,469, general M
global zM zM1 bd                        % Chi-squared RVs, 4 DOF

N=1024;                                 % Number of RVs
n=[1:N]';
b=exp(-.3*(n-.7*N).^2)+.1;              % N means

m=[10;11;12;13]; z=[1.3;.8;.6;.4;170];

M=length(z);
zd=diff(z);
if(any(zd(1:M-2)>0)) pdf=0, keyboard, end
zM=z(M);
zM1=z(M-1);
if(zM>(N+1-M)*zM1) pdf=0, keyboard, end

nd=setdiff(n,m);    % n ~= m(1),...,m(M-1)
bd=b(nd);

options=optimset('tolx',1e-15);
[Ls,fv,ex,ou]=fzero(@Lam1,0,options);
saddlepoint=Ls

ap=z(1:M-1)./b(m);                      % Leading PDFs
pm_log=-sum(ap)-sum(log(b(m)))+sum(log(ap));

as=zM1./bd-zM1*Ls;
Lam=-zM*Ls+2*sum(log(zM1./bd))+sum(Lcs4z(as));
Lam2=zM1^2*sum(Lcs4z2(as));
Lam3=-zM1^3*sum(Lcs4z3(as));
Lam4=zM1^4*sum(Lcs4z4(as));
spa0_log=pm_log+Lam-.5*log(2*pi*Lam2)
ct=.125*Lam4/Lam2^2-5/24*Lam3^2/Lam2^3;
spa1_log=spa0_log+log(1+ct)
error_factor=.5*zM*abs(Lam3)/Lam2^2
keyboard

function w = Lam1(L)    % Lambda'(L)
global zM zM1 bd
w=-zM-zM1*sum(Lcs4z1(zM1./bd-zM1*L));
```

Table 5. Program for Example 4, Six DOF

```
function spa_M_log_4_cs6              % TR 11,469, general M
global zM zM1 bd                      % Chi-squared RVs, 6 DOF

N=1024;                               % Number of RVs
n=[1:N]';
b=exp(-.3*(n-.7*N).^2)+.1;            % N means

m=[10;11;12;13]; z=[1.3;.8;.6;.4;230];

M=length(z);
zd=diff(z);
if(any(zd(1:M-2)>0)) pdf=0, keyboard, end
zM=z(M);
zM1=z(M-1);
if(zM>(N+1-M)*zM1) pdf=0, keyboard, end

nd=setdiff(n,m);     % n ~= m(1),...,m(M-1)
bd=b(nd);

options=optimset('tolx',1e-15);
[Ls,fv,ex,ou]=fzero(@Lam1,0,options);
saddlepoint=Ls

ap=z(1:M-1)./b(m);                    % Leading PDFs
pm_log=-sum(ap)-sum(log(b(m)))+2*sum(log(ap))-(M-1)*log(2);

as=zM1./bd-zM1*Ls;
Lam=-zM*Ls+3*sum(log(zM1./bd))+sum(Lcs6z(as));
Lam2=zM1^2*sum(Lcs6z2(as));
Lam3=-zM1^3*sum(Lcs6z3(as));
Lam4=zM1^4*sum(Lcs6z4(as));
spa0_log=pm_log+Lam-.5*log(2*pi*Lam2)
ct=.125*Lam4/Lam2^2-5/24*Lam3^2/Lam2^3;
spa1_log=spa0_log+log(1+ct)
error_factor=.5*zM*abs(Lam3)/Lam2^2
keyboard

function w = Lam1(L)     % Lambda'(L)
global zM zM1 bd
w=-zM-zM1*sum(Lcs6z1(zM1./bd-zM1*L));
```

Table 6. Program for Example 4, Eight DOF

```
function spa_M_log_4_cs8                    % TR 11,469, general M
global zM zM1 bd                            % Chi-squared RVs, 8 DOF

N=1024;                                     % Number of RVs
n=[1:N]';
b=exp(-.3*(n-.7*N).^2)+.1;                  % N means

m=[10;11;12;13]; z=[1.3;.8;.6;.4;270];

M=length(z);
zd=diff(z);
if(any(zd(1:M-2)>0)) pdf=0, keyboard, end
zM=z(M);
zM1=z(M-1);
if(zM>(N+1-M)*zM1) pdf=0, keyboard, end

nd=setdiff(n,m);      % n ~= m(1),...,m(M-1)
bd=b(nd);

options=optimset('tolx',1e-15);
[Ls,fv,ex,ou]=fzero(@Lam1,0,options);
saddlepoint=Ls

ap=z(1:M-1)./b(m);                          % Leading PDFs
pm_log=-sum(ap)-sum(log(b(m)))+3*sum(log(ap))-(M-1)*log(6);

as=zM1./bd-zM1*Ls;
Lam=-zM*Ls+4*sum(log(zM1./bd))+sum(Lcs8z(as));
Lam2=zM1^2*sum(Lcs8z2(as));
Lam3=-zM1^3*sum(Lcs8z3(as));
Lam4=zM1^4*sum(Lcs8z4(as));
spa0_log=pm_log+Lam-.5*log(2*pi*Lam2)
ct=.125*Lam4/Lam2^2-5/24*Lam3^2/Lam2^3;
spa1_log=spa0_log+log(1+ct)
error_factor=.5*zM*abs(Lam3)/Lam2^2
keyboard

function w = Lam1(L)     % Lambda'(L)
global zM zM1 bd
w=-zM-zM1*sum(Lcs8z1(zM1./bd-zM1*L));
```

SUMMARY

Equations have been derived for the standard and first-order corrected SPA to the combined probability and joint PDF of the $M-1$ largest RVs of a set of N independent RVs and the sum of the remainder. These results have been applied to six different examples of RVs, including Exponential, Gaussian-Squared, Gaussian, and Chi-Squared RVs of various DOF. Detailed investigations of the necessary auxiliary functions and their singularities have been conducted, and highly accurate programs have been generated for their efficient calculation. Most of the results have 13 to 14 decimal digits of accuracy.

It has been observed numerically that the SPA is very accurate for evaluating the joint PDF of interest here; in fact, the first-order correction term is virtually 1 in most cases. The reason for this is that the integral in equation (3) pertains to the last RV, which is the sum r of the remainder. For a large number N of RVs, and for M not too large, the Central Limit Theorem says that r (the sum of $N+1-M$ RVs) tends to a Gaussian RV. But the SPA itself is *exact* for Gaussian RVs. Therefore, since r is nearly Gaussian, it is expected that the SPA should be quite accurate. This is indeed the case.

REFERENCES

1. A. H. Nuttall, "Joint Probability Density Function of Selected Order Statistics and the Sum of the Remaining Random Variables," NUWC-NPT Technical Report 11,345, Naval Undersea Warfare Center Division, Newport, RI, 15 January 2002.

2. A. H. Nuttall, "Joint Probability Density Function of Selected Order Statistics and the Sum of the Remainder as Applied to Arbitrary Independent Random Variables," NUWC-NPT Technical Report 11,469, Naval Undersea Warfare Center Division, Newport, RI, 6 November 2003.

APPENDIX A
ALTERNATIVE POWER SERIES FOR log[(1 – exp(–z))/z]

The function $L(z)$ of interest and its first four derivatives were given in equations (12) and (16), namely,

$$L(z) = \log\left(\frac{1-\exp(-z)}{z}\right) \quad for\ z \neq 0, \quad L(0) = 0, \tag{A-1}$$

and

$$\begin{aligned}
L'(z) &= -\frac{1}{z} + \frac{1}{E-1}, \quad E \equiv \exp(z), \\
L''(z) &= \frac{1}{z^2} - \frac{E}{(E-1)^2}, \\
L'''(z) &= -\frac{2}{z^3} + \frac{E(E+1)}{(E-1)^3}, \\
L''''(z) &= \frac{6}{z^4} - \frac{E(E^2+4E+1)}{(E-1)^4},
\end{aligned} \tag{A-2}$$

where z is complex. The values of the functions in equation (A-2), at $z = 0$, are to be taken as the limits as z approaches zero. All of these equations have numerical problems as z approaches zero, and the problem becomes most severe for the highest-order derivative. The required differences of large numbers, of approximately equal size, cause losses of significance in the final results near $z = 0$, if equations (A-1) and (A-2) are used in their current forms.

An alternative approach is to replace these equations with their power series expansions about $z = 0$. Questions then arise as to how many terms should be used in each power series, and at what values of $|z|$ the switches should be made, from the given functional forms above, to the appropriate power series. The answers to these questions hinge on the desired accuracy, both absolute and relative, and the amount of computer effort to be expended on the power series equivalents.

The approach to this problem will be illustrated with respect to function $L(z)$ in equation (A-1). First, "exact" values of $L(z)$ were obtained directly from equation (A-1) (at any nonzero z of interest) by means of variable precision arithmetic using 50 digits of significance. Then, a (nonzero) value of radius $r = |z|$ was specified and $L(z)$ was evaluated via equation (A-1) at 128 equally spaced points on a circle of radius r in the complex z-plane, using standard double precision (64 bits). The maximum absolute and relative errors of these latter calculations (relative to the "exact" 50-digit results) were then evaluated and plotted as one point on each of the curves labeled "L" in figure A-1. The radius r was then varied from 0.05 to 1.2 in increments of 0.05 and the complete "L" curves, both absolute and relative in figure A-1, were swept out. These error curves basically decrease with radius r because the distance from the problematical point $z = 0$ is increasing.

Next, the power series expansion of $L(z)$ to 8 terms, namely up to z^8, was determined. For radius r, this power series in z was then evaluated in standard double precision at the same 128 points as above, and the maximum absolute and relative errors (relative to "exact") were computed and plotted in figure A-1 with the label "z^8". As radius r was increased, the errors basically increased because the power series is most accurate near the origin of the complex z-plane. This procedure was repeated for the power series expansions using 10, 12, 14, and 16 terms, and the corresponding absolute and relative errors are indicated in figure A-1.

From the plots in figure A-1, it is now possible to determine where to make the switch from the functional form in equation (A-1) to one of the various power series indicated. For present purposes with function $L(z)$ in equation (A-1), it was decided to make the switch at radius $r =$ 0.56 and to use the power series with terms up to z^{12} for $|z| < 0.56$. Figure A-1 indicates that the relative error is then better than 10^{-15} for all complex z. A larger switch-radius would require a larger power series, but the improvement in relative error would be minimal.

Similar results for the four derivative functions in equation (A-2) are presented in figures A-2 through A-5, respectively. The quantity $L^{(p)}(z)$ is denoted by L_p in the figures. It was decided to use 13 terms for $L'(z)$ below radius 0.63, 16 terms for $L''(z)$ below radius 0.9, 23 terms for $L'''(z)$ below radius 1.42, and 24 terms for $L''''(z)$ below radius 1.45. The errors are getting progressively poorer for the higher-order derivative functions. However, the worst relative error for $L''''(z)$ still maintains about 12.7 decimals, according to figure A-5. This bound is applicable for all complex z.

The slow decays of the L-curves with radius r discourage the use of higher-order series expansions. However, if larger errors can be tolerated, figures A-1 through A-5 indicate the trade-offs available to a user. For example, if 20 terms are acceptable for the power series for $L''''(z)$, the relative error would be increased to about 12.4 decimals and the switch radius would be changed to 1.2.

All of these functions have poles at $z = i2\pi n \;\; for \; n = \pm 1, \pm 2, \; \ldots$. This non-analytic behavior means that the power series expansions about the origin $z = 0$ cannot possibly converge for $|z| > 2\pi$, because that is the location of the poles closest to the origin. Furthermore, one cannot get too close to $|z| = 2\pi$ because the convergence would then be so slow as to be futile. Also observe that the pole of $L''''(z)$ at $z = \pm i2\pi$ is of *fourth* order; that singularity makes for very slow convergence if $|z|$ is allowed to get anywhere near 2π.

Programs for all the functions in equations (A-1) and (A-2) are presented in tables A-1 through A-5. These MATLAB programs are vectorized for any size matrix z and can handle all complex values of z.

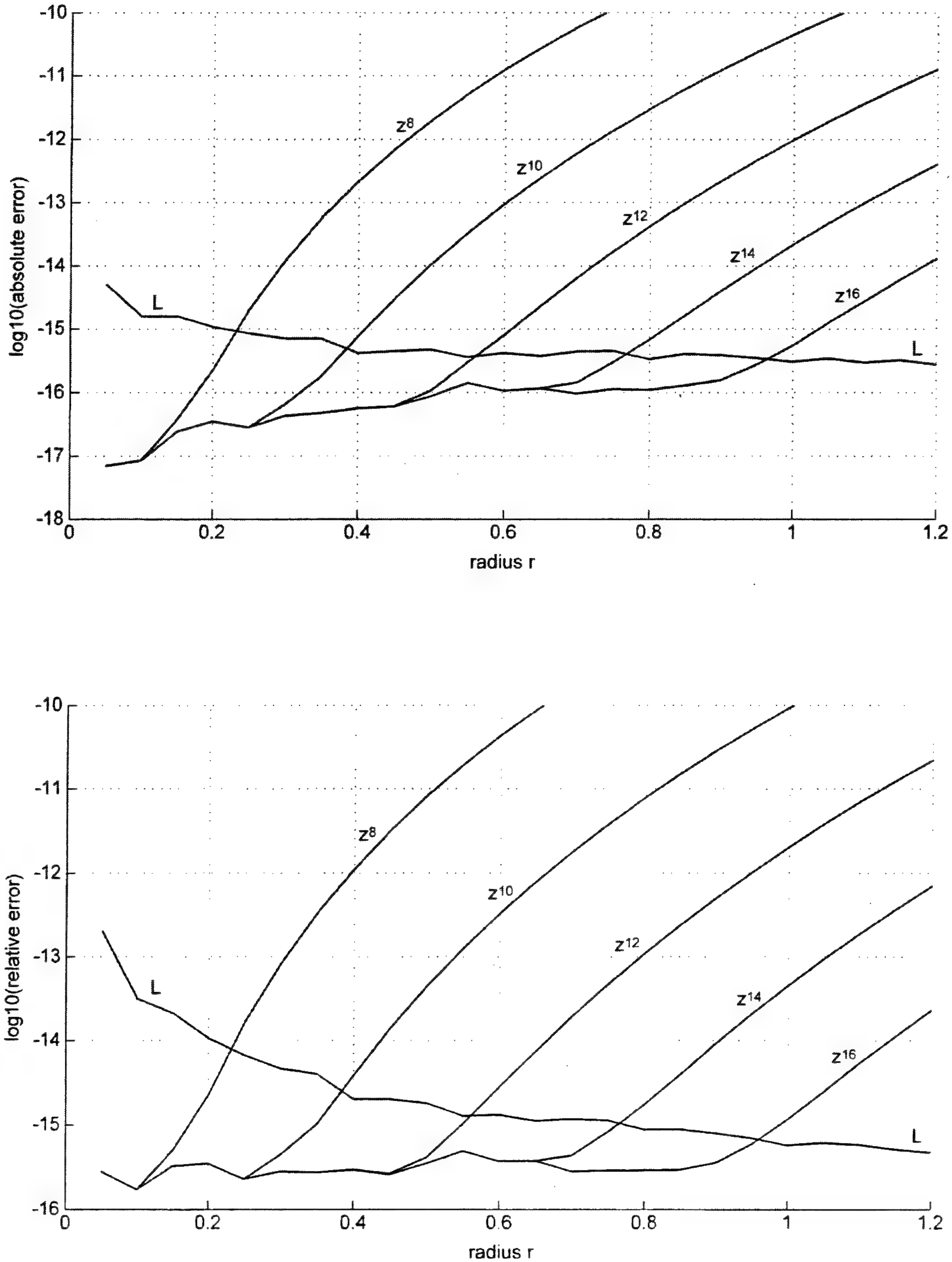


Figure A-1. Errors for L(z) in Equation (A-1)

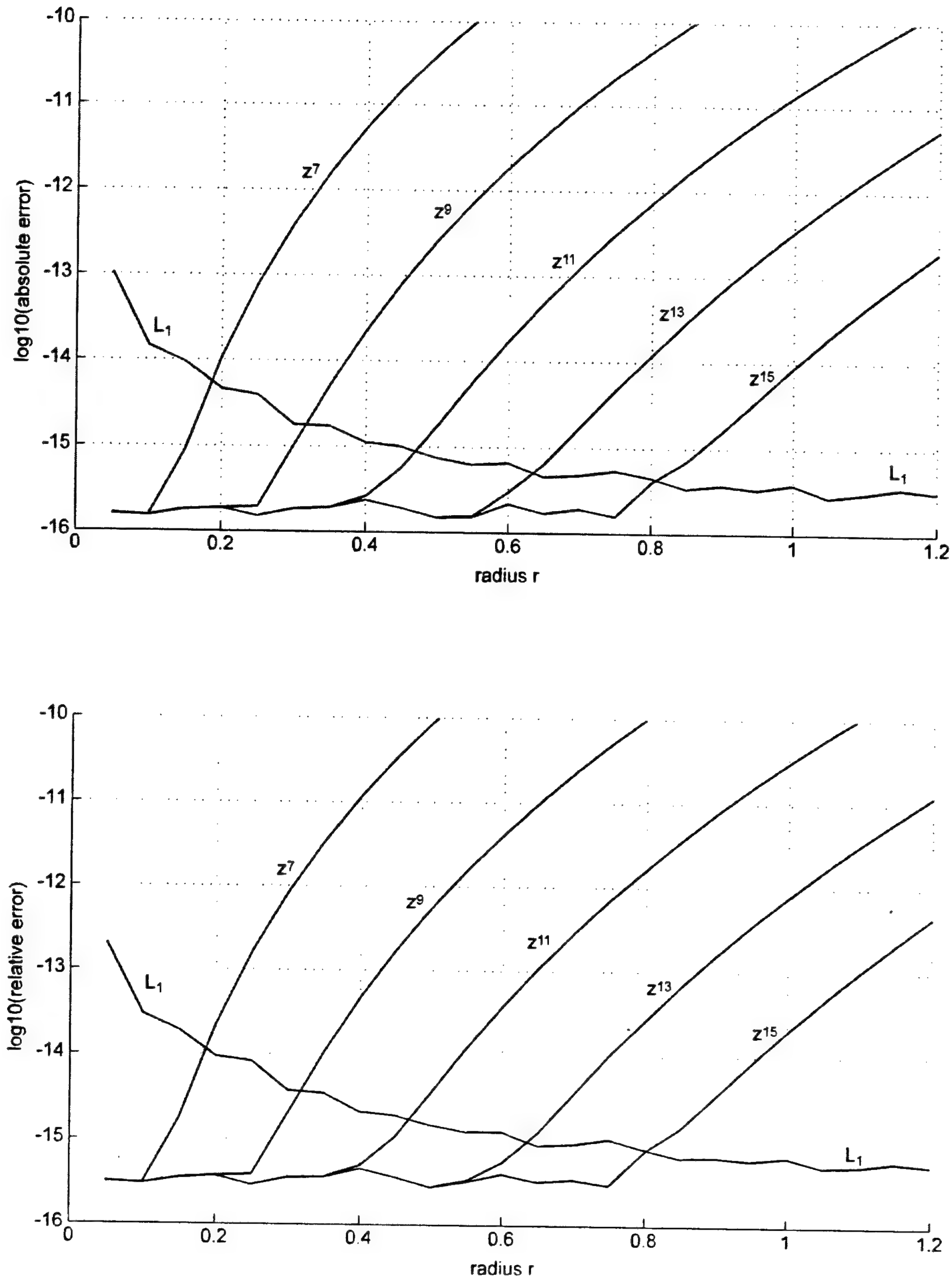


Figure A-2. Errors for L'(z) in Equation (A-2)

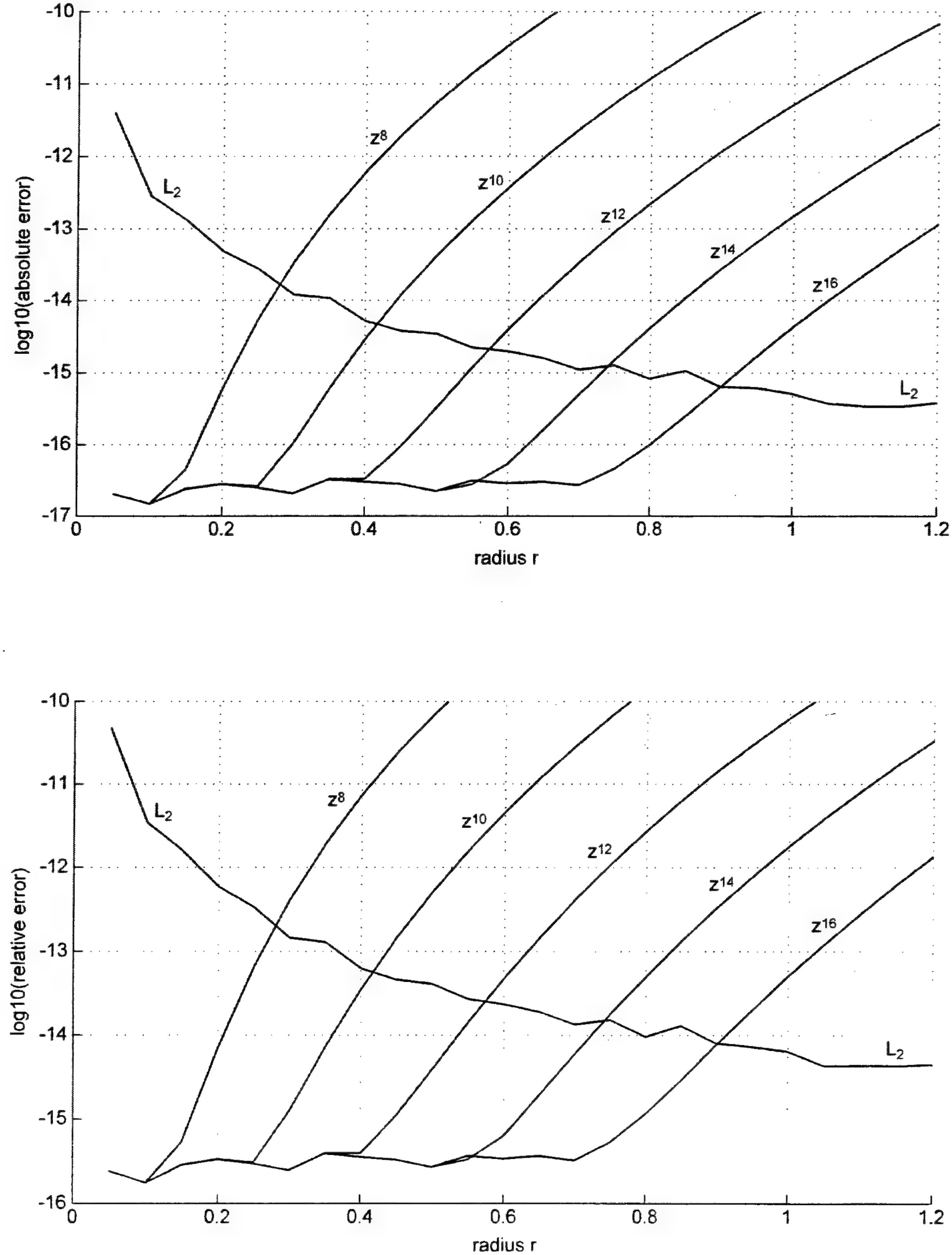


Figure A-3. Errors for L"(z) in Equation (A-2)

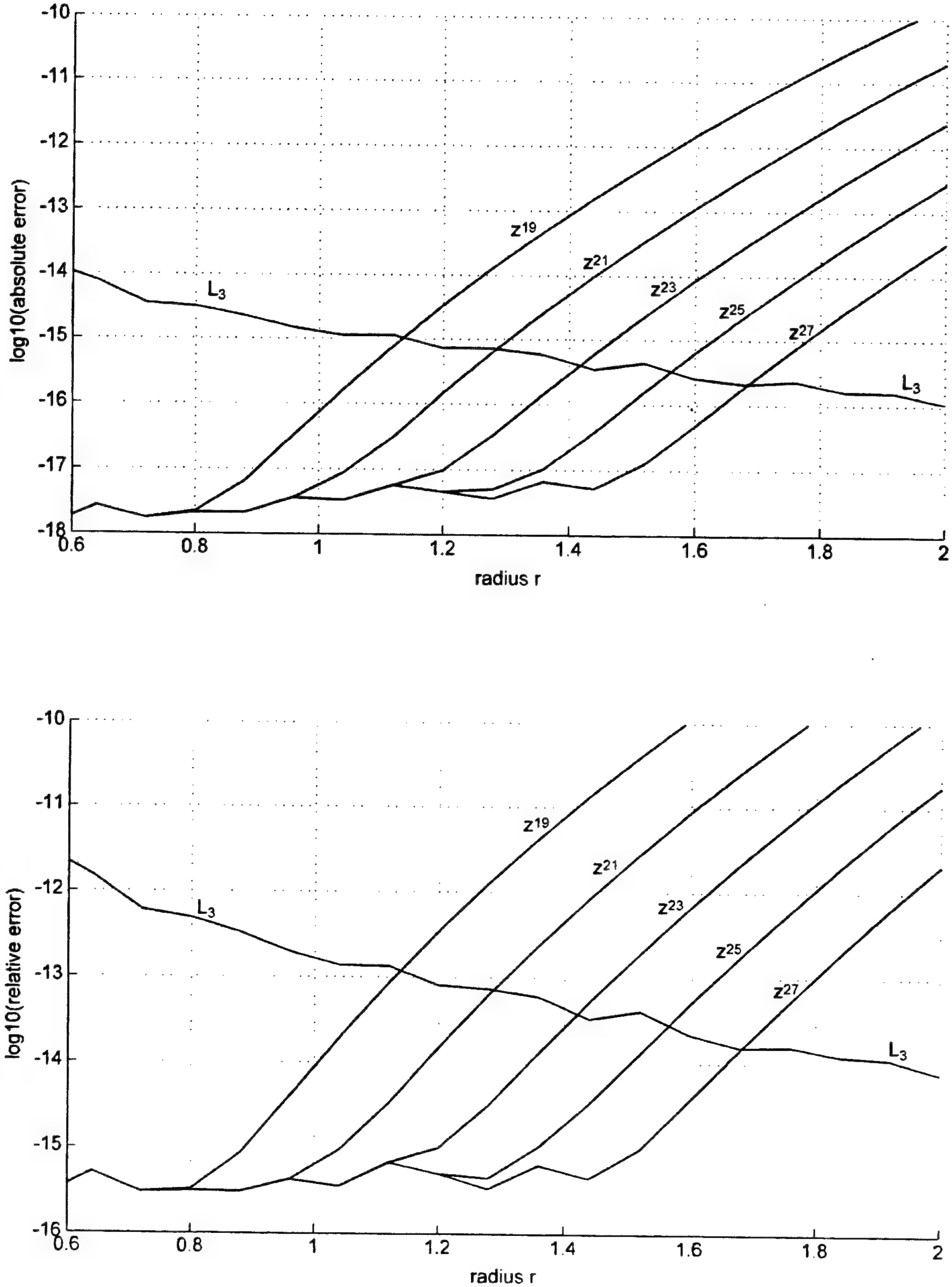


Figure A-4. Errors for L'''(z) in Equation (A-2)

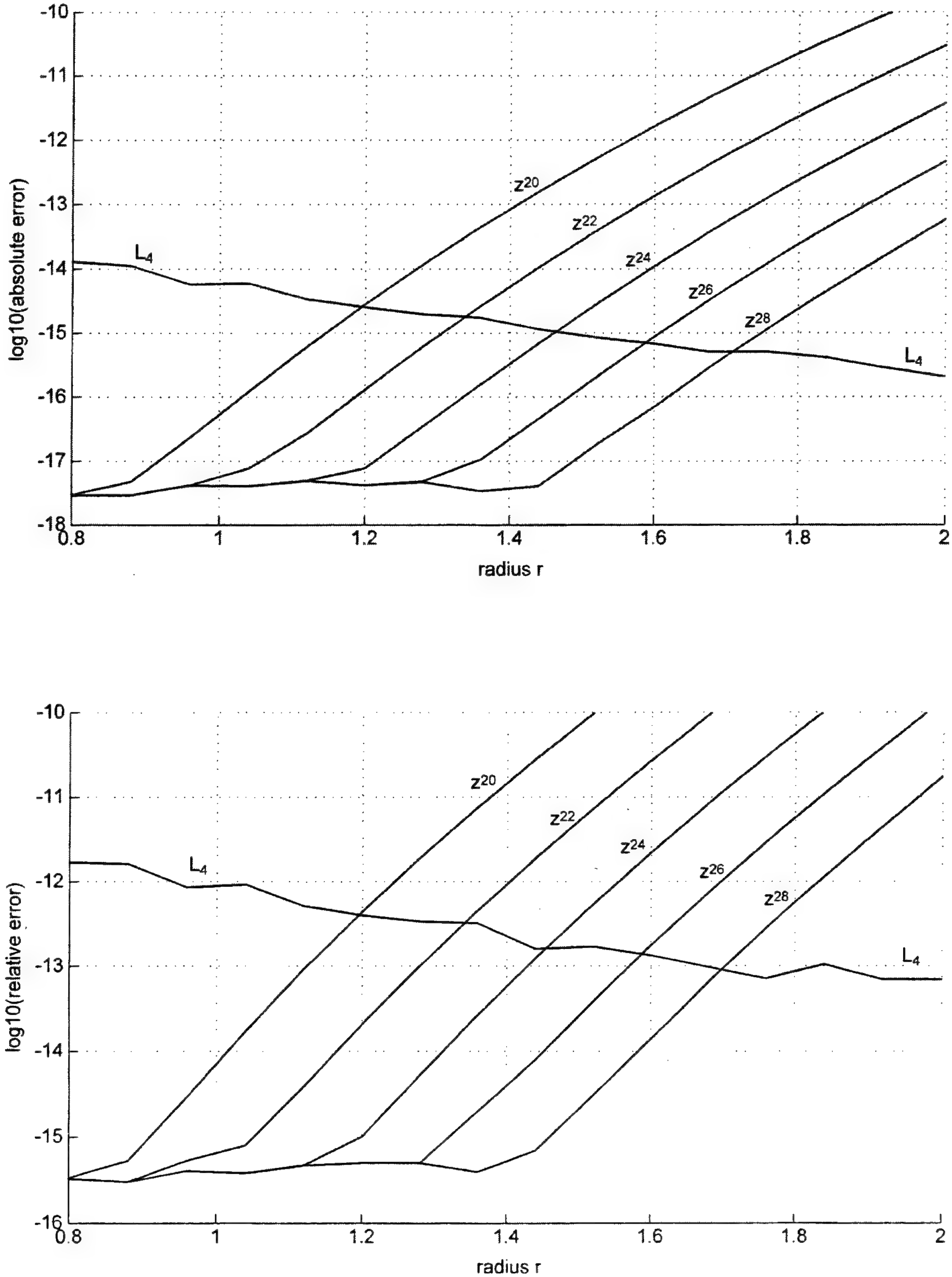


Figure A-5. Errors for L''''(z) in Equation (A-2)

Table A-1. Program for L(z) in Equation (A-1)

```
function w = Lcs2z(z)    % log([1-exp(-z)]/z)
s=size(z); z=z(:);       % for all complex z
w=zeros(size(z));        % and all size(z).
% Principal value log is desired.  The branch lines are at:
% z = lambertw(n,exp(1/p)/p)-1/p for real p > 0; n=+-1, +-2,...
% Branch points of Log(...) are at z = i 2 pi n, n=+-1, +-2,...

k1=find(real(z)<=-37);
if(~isempty(k1))
    y=z(k1);
    t=-y-log(-y);   % Not necessarily the principal value for w.
    b=imag(t);
    b=mod(b,2*pi);
    k=find(b>pi);
    b(k)=b(k)-2*pi;
    w(k1)=real(t)+b*i;  % Principal value for w.
end

k2=find(abs(z)<=.56);    % Radius 0.56
if(~isempty(k2))
    y=z(k2); x=y.*y;
    a1=-1/2; a2=1/24; a4=-1/2880; a6=1/181440; a8=-1/9676800;
    a10=2.087675698786810e-9; a12=-4.403491782239578e-11;
    w(k2)=a1*y+x.*(a2+x.*(a4+x.*(a6+x.*(a8+x.*(a10+x.*a12)))));
end

k3=setdiff([1:length(z)]',union(k1,k2));
if(~isempty(k3))
    y=z(k3);
    w(k3)=log((1-exp(-y))./y);
end

w=reshape(w,s);
```

Table A-2. Program for L'(z) in Equation (A-2)

```
function w = Lcs2z1(z)  % -1/z+1/(exp(z)-1)
s=size(z); z=z(:);      % for all complex z
w=zeros(size(z));       % and all size(z).
% Function has poles at i 2 pi n for n=+-1, +-2,...

k1=find(real(z)>=41);
if(~isempty(k1))
    w(k1)=-1./z(k1);
end

k2=find(abs(z)<=.63);    % Radius 0.63
if(~isempty(k2))
    y=z(k2); x=y.*y;
    a0=-1/2; a1=1/12; a3=-1/720; a5=1/30240; a7=-1/1209600;
    a9=1/47900160; a11=-5.284190138687493e-10;
    a13=1.338253653068468e-11;
    w(k2)=a0+y.*(a1+x.*(a3+x.*(a5+x.*(a7+x.*...
    (a9+x.*(a11+x.*a13))))));
end

k3=setdiff([1:length(z)]',union(k1,k2));
if(~isempty(k3))
    y=z(k3);
    w(k3)=-1./y+1./(exp(y)-1);
end

w=reshape(w,s);
```

Table A-3. Program for L"(z) in Equation (A-2)

```
function w = Lcs2z2(z)  % 1/z^2-E/(E-1)^2; E=exp(z)
s=size(z); z=z(:);      % for all complex z
w=zeros(size(z));       % and all size(z).
% Function has poles at i 2 pi n for n=+-1, +-2,...

k1=find(real(z)>=45);
if(~isempty(k1))
    w(k1)=1./z(k1).^2;
end

k2=find(abs(z)<=.9);     % Radius 0.9
if(~isempty(k2))
    y=z(k2); x=y.*y;
    a0=1/12; a2=-1/240; a4=1/6048; a6=-1/172800; a8=1/5322240;
    a10=-691/118879488000; a12=1/5748019200;
    a14=-3617/711374856192000; a16=1.459630549567234e-13;
    w(k2)=a0+x.*(a2+x.*(a4+x.*(a6+x.*(a8+x.*...
    (a10+x.*(a12+x.*(a14+x.*a16)))))));
end

k3=setdiff([1:length(z)]',union(k1,k2));
if(~isempty(k3))
    y=z(k3); E=exp(y);
    w(k3)=1./y.^2-E./(E-1).^2;
end

w=reshape(w,s);
```

Table A-4. Program for L'''(z) in Equation (A-2)

```
function w = Lcs2z3(z)   % -2/z^3+E(E+1)/(E-1)^3; E=exp(z)
s=size(z); z=z(:);       % for all complex z
w=zeros(size(z));        % and all size(z).
% Function has poles at i 2 pi n for n=+-1, +-2,...

k1=find(real(z)>=49);
if(~isempty(k1))
    w(k1)=-2./z(k1).^3;
end

k2=find(abs(z)<=1.42);   % Radius 1.42
if(~isempty(k2))
    y=z(k2); x=y.*y;
    a1=-1/120; a3=1/1512; a5=-1/28800; a7=1/665280;
    a9=-691/11887948800; a11=1/479001600; a13=-7.118328622277424e-11;
    a15=2.335408879307574e-12; a17=-7.438050949068572e-14;
    a19=2.313781187911296e-15; a21=-7.060959131021136e-17;
    a23=2.120824223777681e-18;
    w(k2)=y.*(a1+x.*(a3+x.*(a5+x.*(a7+x.*(a9+x.*(a11...
    +x.*(a13+x.*(a15+x.*(a17+x.*(a19+x.*(a21+x.*a23)))))))))));
end

k3=setdiff([1:length(z)]',union(k1,k2));
if(~isempty(k3))
    y=z(k3); E=exp(y);
    w(k3)=-2./y.^3+E.*(E+1)./(E-1).^3;
end

w=reshape(w,s);
```

Table A-5. Program for L""(z) in Equation (A-2)

```
function w = Lcs2z4(z)  % 6/z^4-E(E^2+4E+1)/(E-1)^4; E=exp(z)
s=size(z); z=z(:);      % for all complex z
w=zeros(size(z));       % and all size(z).
% Function has poles at i 2 pi n for n = +-1, +-2,...

k1=find(real(z)>=52);
if(~isempty(k1))
    w(k1)=6./z(k1).^4;
end

k2=find(abs(z)<=1.45);  % Radius 1.45
if(~isempty(k2))
    y=z(k2); x=y.*y;
    a0=-1/120; a2=1/504; a4=-1/5760; a6=1/95040;
    a8=-691/1320883200; a10=1/43545600; a12=-9.253827208960651e-10;
    a14=3.503113318961361e-11; a16=-1.264468661341657e-12;
    a18=4.396184257031463e-14; a20=-1.482801417514439e-15;
    a22=4.877895714688665e-17; a24=-1.571342308445089e-18;
    w(k2)=a0+x.*(a2+x.*(a4+x.*(a6+x.*(a8+x.*(a10+x.*(a12...

    +x.*(a14+x.*(a16+x.*(a18+x.*(a20+x.*(a22+x.*a24)))))))))));
end

k3=setdiff([1:length(z)]',union(k1,k2));
if(~isempty(k3))
    y=z(k3); E=exp(y);
    w(k3)=6./y.^4-E.*(E.^2+4*E+1)./(E-1).^4;
end

w=reshape(w,s);
```

APPENDIX B
ALTERNATIVE POWER SERIES FOR $\log[\mathrm{erf}(\sqrt{z})/\sqrt{z}]$

The function $L(z)$ of interest and its first four derivatives were given in equations (19) and (23), namely,

$$L(z) = \log\left(\frac{\mathrm{erf}(\sqrt{z})}{\sqrt{z}}\right) \text{ for } z \neq 0, \quad L(0) = \log(2/\sqrt{\pi}), \tag{B-1}$$

and

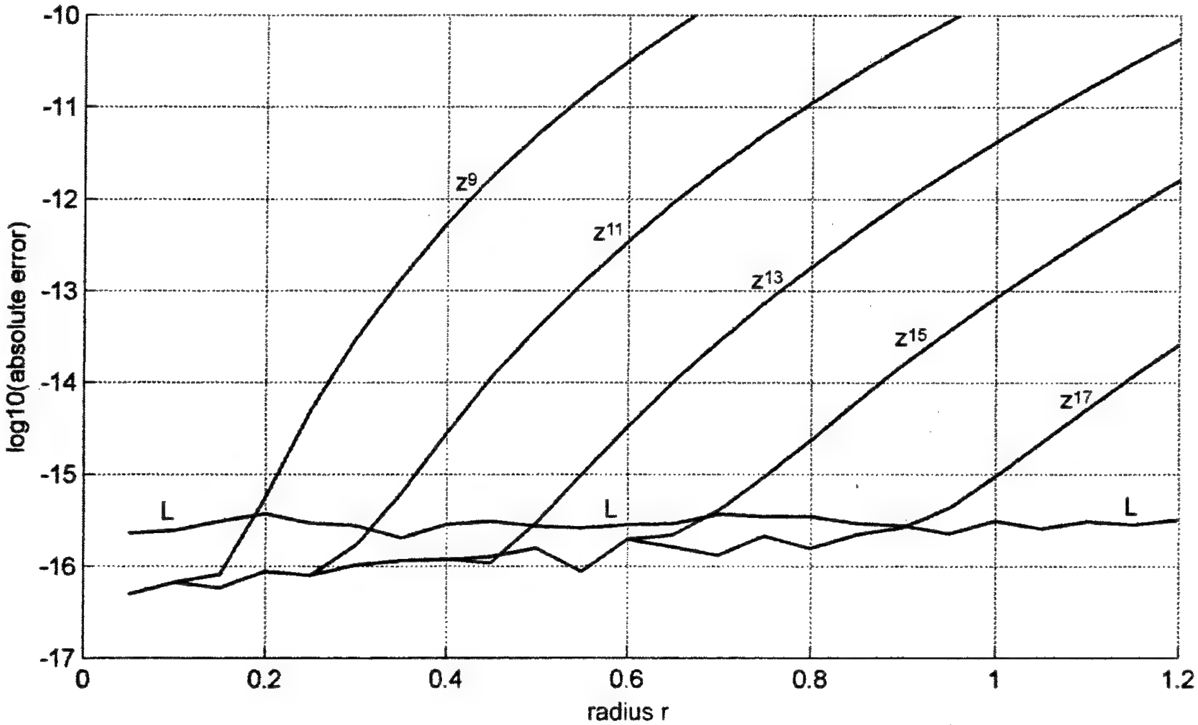
$$\begin{aligned}
L'(z) &= -\frac{1}{2z} + G, \quad G \equiv \frac{\exp(-z)}{\sqrt{\pi z}\,\mathrm{erf}(\sqrt{z})}, \\
L''(z) &= \frac{1}{2z^2} - \left(1 + \frac{0.5}{z}\right)G - G^2, \\
L'''(z) &= -\frac{1}{z^3} + \left(1 + \frac{1}{z} + \frac{0.75}{z^2}\right)G + \left(3 + \frac{1.5}{z}\right)G^2 + 2G^3, \\
L''''(z) &= \frac{3}{z^4} - \left(1 + \frac{1.5}{z} + \frac{2.25}{z^2} + \frac{1.875}{z^3}\right)G - \left(7 + \frac{7}{z} + \frac{3.75}{z^2}\right)G^2 - \left(12 + \frac{6}{z}\right)G^3 - 6G^4,
\end{aligned} \tag{B-2}$$

where z is complex. The values of the functions in equation (B-2) at $z = 0$ are to be taken as the limits as z approaches zero. All of these equations have numerical problems as z approaches zero, and the problem becomes most severe for the highest-order derivative. The required differences of large numbers, of approximately equal size, causes loss of significance in the final results near $z = 0$, if equations (B-1) and (B-2) are used in their current forms.

The method of circumventing this numerical difficulty is identical to that presented in appendix A, namely, replace these functions by their power series representations for z near 0. The results for the absolute and relative errors for the five functions above are presented in figures B-1 through B-5, respectively. For the L-function error in figure B-1, it was decided to use equation (B-1), as is, for all z. On the other hand, it was decided to use 6 terms in the power series for $L'(z)$ below radius 0.06, 16 terms for $L''(z)$ below radius 0.71, 18 terms for $L'''(z)$ below radius 0.87, and 22 terms for $L''''(z)$ below radius 1.08. Thus, for example, from figure B-5, the relative error for $L''''(z)$ is 12.1 decimals for all complex z.

There are bumps in some of the relative error curves. These are due to the exact function having a zero for a particular value of z. For example, $L(z)$ has a zero at $z = 0.3812$, $L'''(z)$ has a zero at $z = -1.4585$, and $L''''(z)$ has a zero at $z = 0.9087$. When the relative error is computed at a point near these zeros, these near-zero calculations of the denominator manifest themselves as bumps in the relative error curve, and should be ignored.

MATLAB programs for all five functions in equations (B-1) and (B-2) are available from the author upon request.



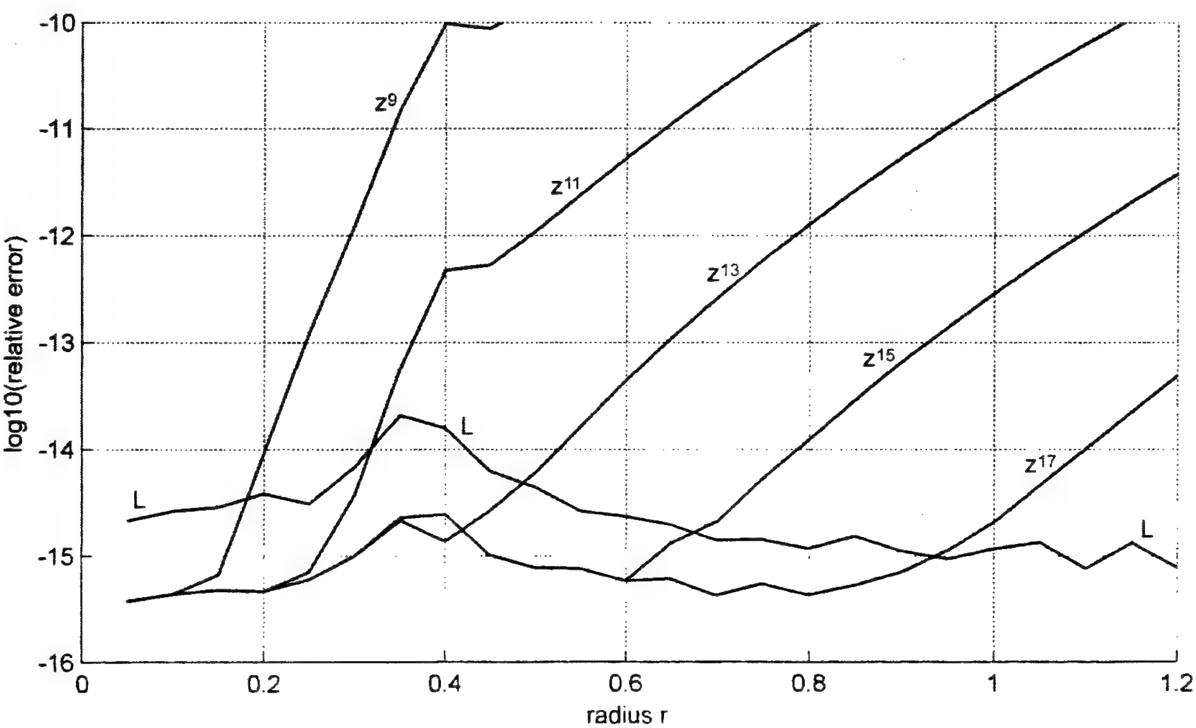
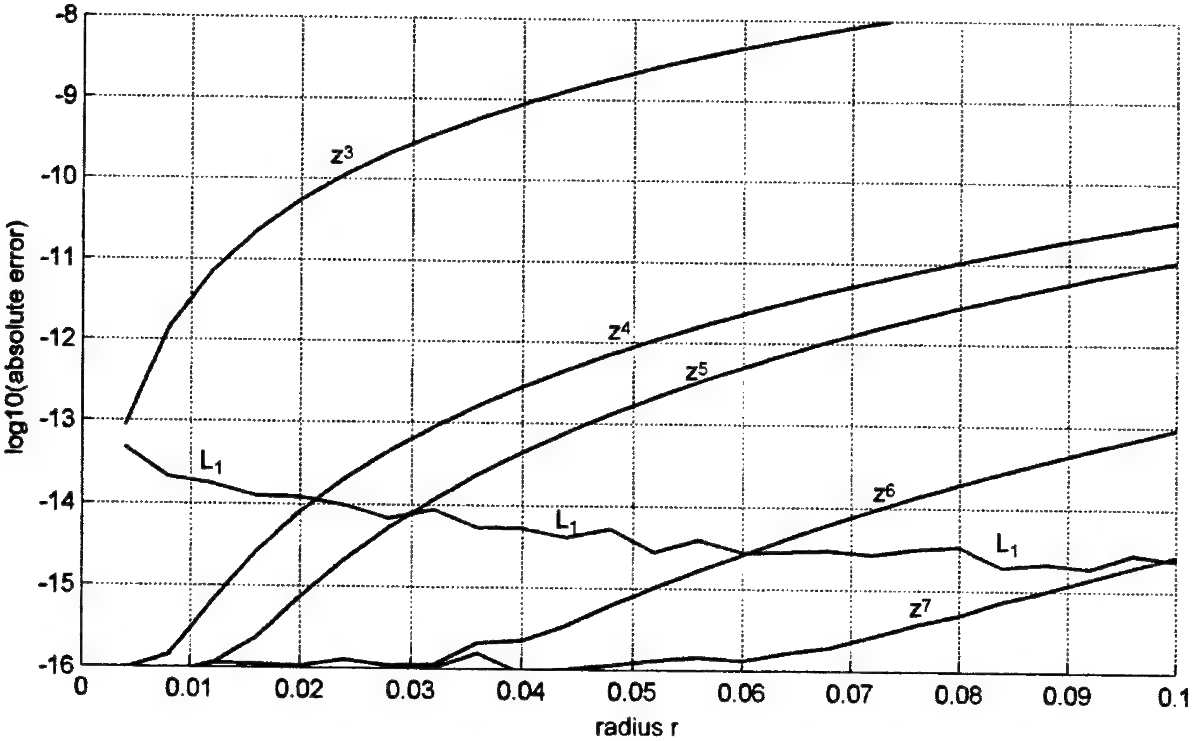


Figure B-1. Errors for L(z) in Equation (B-1)



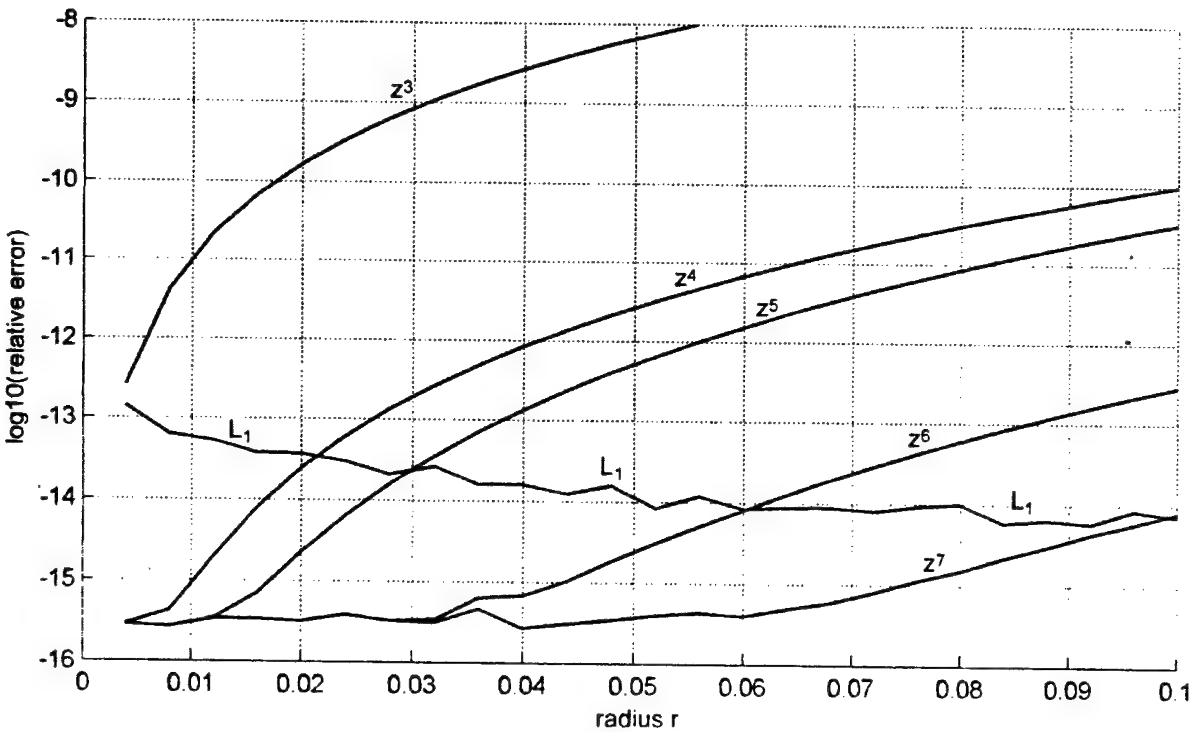


Figure B-2. Errors for $L'(z)$ in Equation (B-2)

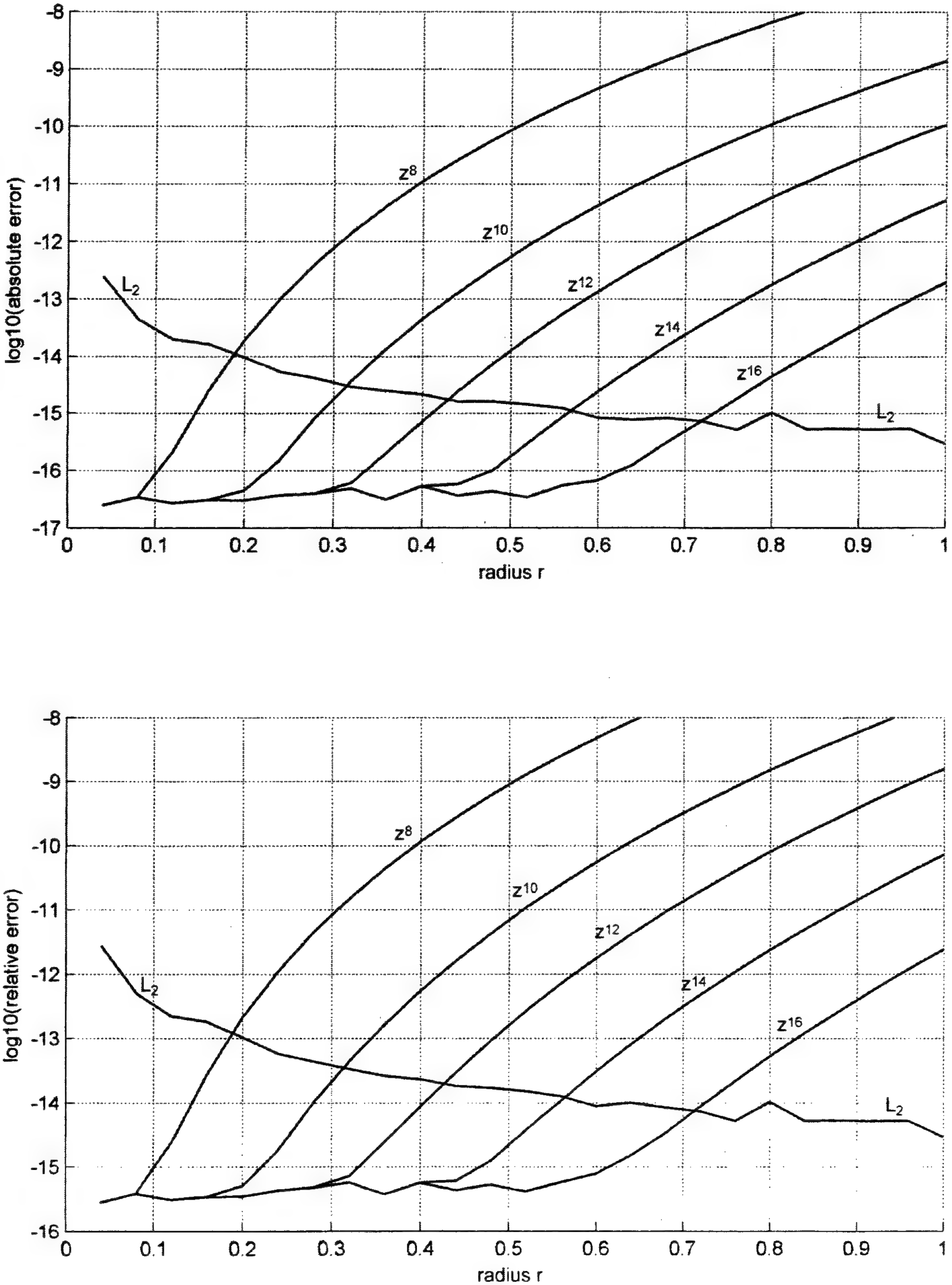


Figure B-3. Errors for L"(z) in Equation (B-2)

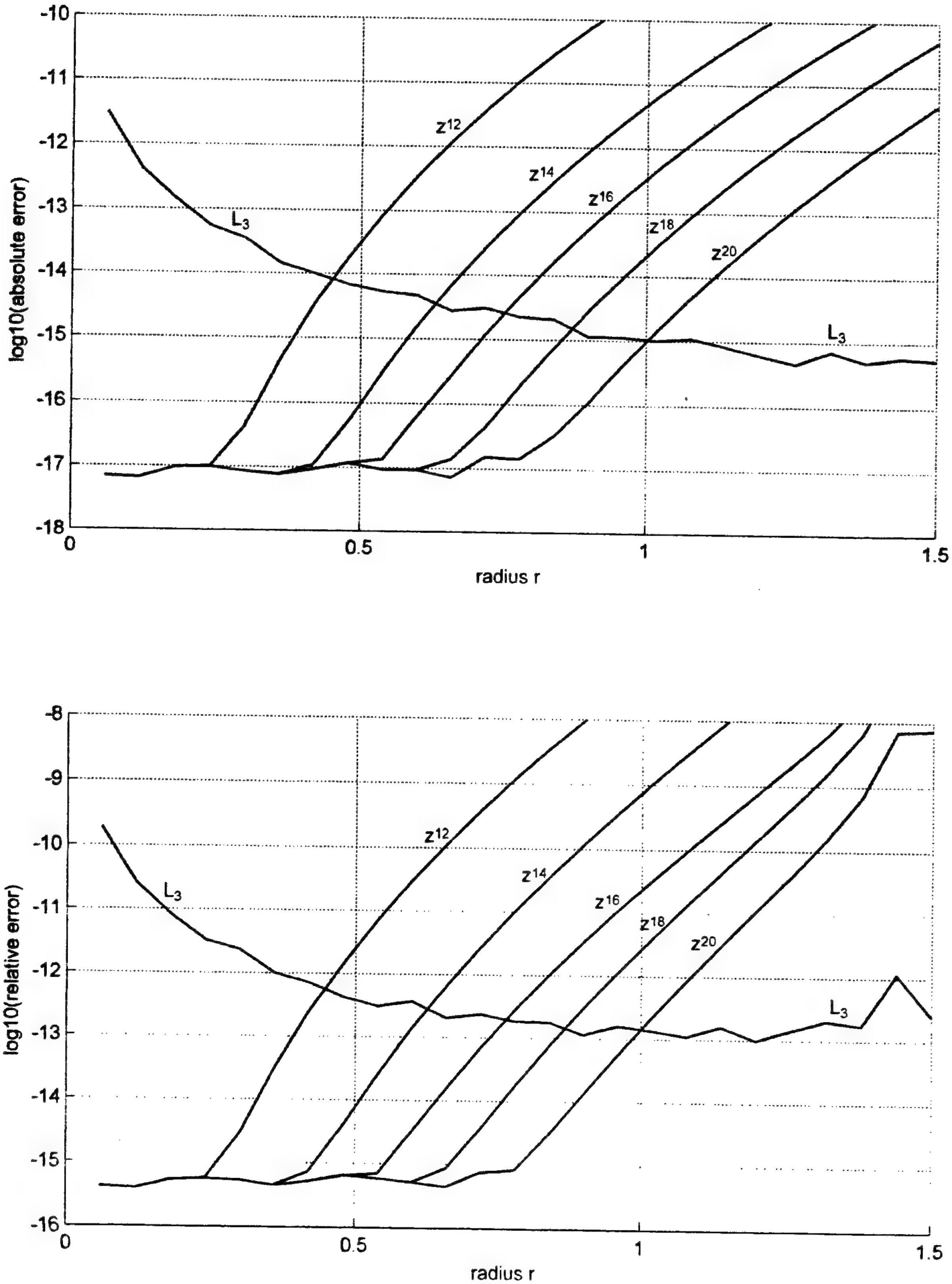
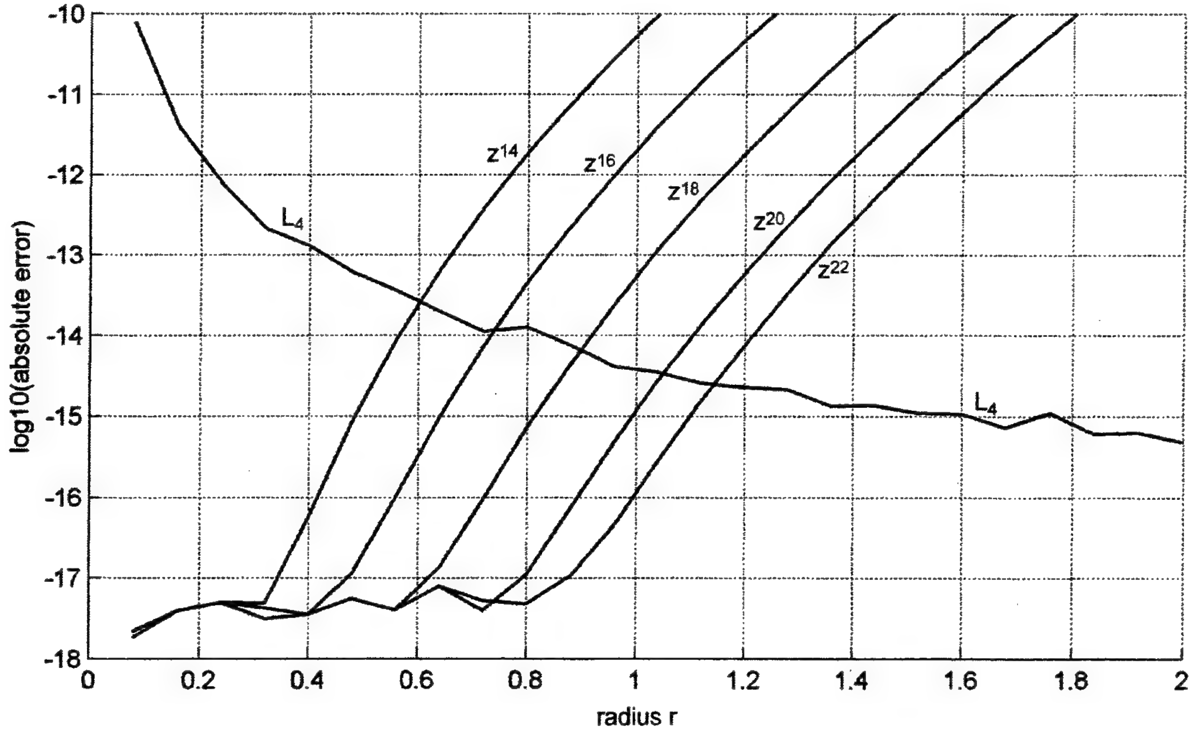


Figure B-4. Errors for L'''(z) in Equation (B-2)



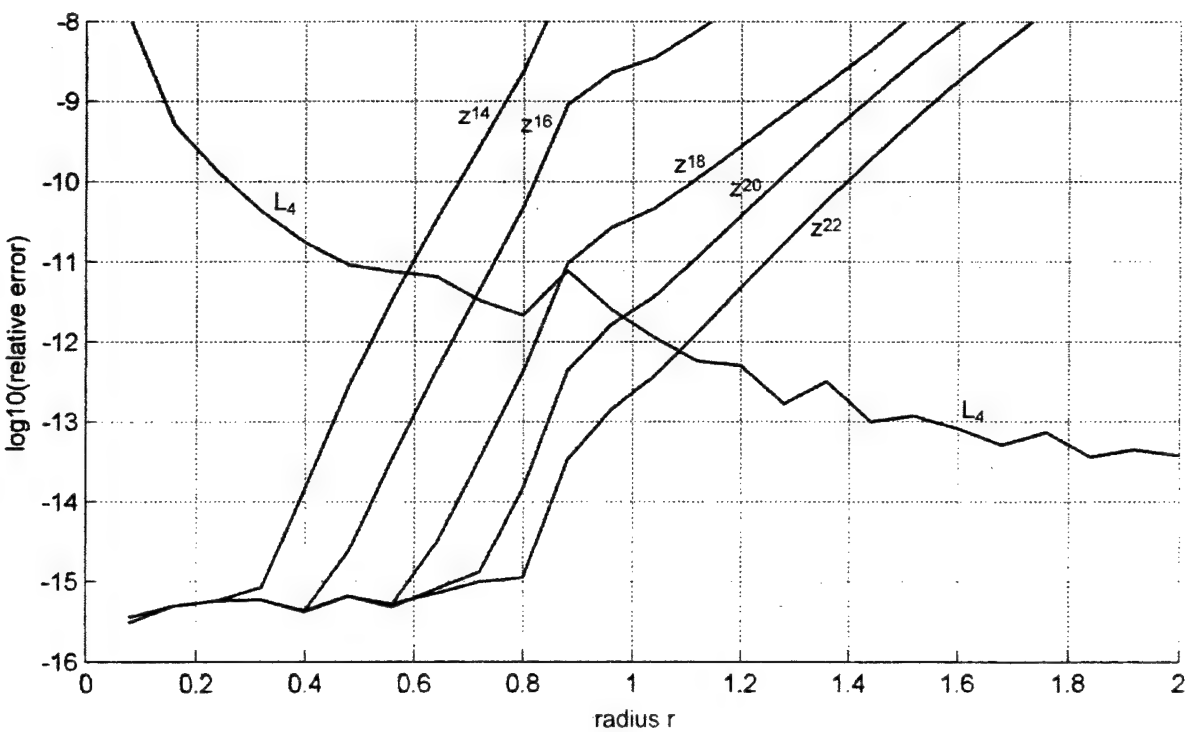


Figure B-5. Errors for L""(z) in Equation (B-2)

APPENDIX C
ALTERNATIVE POWER SERIES FOR CHI-SQUARED RANDOM VARIABLES

For $k = 1$, that is, for a chi-squared RV with $2k + 2 = 4$ DOF, the function $L(z)$ and its first four derivatives were given in equation (38), namely,

$$
\begin{aligned}
L(z) &= \log\left(\frac{1-\exp(-z)\,(1+z)}{z^2}\right) \quad \text{for } z \neq 0, \quad L(0) = -\log(2), \\
L'(z) &= -\frac{2}{z} + \frac{z}{E-1-z}, \quad E \equiv \exp(z), \\
L''(z) &= \frac{2}{z^2} + \frac{E\,(1-z)-1}{(E-1-z)^2}, \\
L'''(z) &= -\frac{4}{z^3} + \frac{E^2\,(-2+z) + E\,(4-z+z^2) - 2}{(E-1-z)^3}, \\
L''''(z) &= \frac{12}{z^4} + \frac{E^3\,(3-z) + 4\,E^2\,(-3+2z-z^2) + E\,(15-7z+z^2-z^3) - 6}{(E-1-z)^4},
\end{aligned}
\tag{C-1}
$$

where z is complex. The values of the functions in equation (C-1), at $z = 0$, are to be taken as the limits as z approaches zero. All of these equations have numerical problems as z approaches zero, and the problem becomes most severe for the highest-order derivative. The required differences of large numbers, of approximately equal size, causes losses of significance in the final results near $z = 0$, if the results in equation (C-1) are used in their current forms.

The method of circumventing this numerical difficulty is identical to that presented in appendix A, namely, replace these functions by their power series representations for z near 0. The results for the absolute and relative errors, for the five functions above, are presented in figures C-1 through C-5, respectively. It was decided to use 13 terms in the power series for $L(z)$ below radius 0.7, 14 terms for $L'(z)$ below radius 0.9, 15 terms for $L''(z)$ below radius 1.03, 18 terms for $L'''(z)$ below radius 1.27, and 23 terms for $L''''(z)$ below radius 1.84. Thus, for example, from figure C-5, the maximum relative error for $L''''(z)$ is 12 decimals for all complex z.

The relative error curve for $L(z)$ has a bump in the neighborhood of radius 1; this is due to a zero of $L(z)$ at $z = -1$. Also, $L'''(z)$ has a zero at $z = 2.1631$, while $L''''(z)$ has zeros at $z = -1.1756$ and 5.4429; however, the only one of these latter zeros that affects the relative error curve is the bump in the neighborhood of radius 1.2 in the plot of $L''''(z)$.

For $k = 2$, that is, for a chi-squared RV with $2k + 2 = 6$ DOF, the function $L(z)$ and its first four derivatives were given in equation (39), namely

$$
\begin{aligned}
L(z) &= \log\left(\frac{1-\exp(-z)\,(1+z+z^2/2)}{z^3}\right) \quad \text{for } z \neq 0, \quad L(0) = -\log(6), \\
L'(z) &= -\frac{3}{z} + \frac{z^2}{2D}, \quad E \equiv \exp(z), \quad D \equiv E - (1+z+z^2/2), \\
L''(z) &= \frac{3}{z^2} + z\frac{E(2-z)-2-z}{2D^2}, \\
L'''(z) &= -\frac{6}{z^3} + \frac{E^2(4-8z+2z^2)+E(-8+8z+8z^2-2z^3+z^4)+4-6z^2-2z^3}{4D^3}, \\
L''''(z) &= \frac{18}{z^4} + \frac{1}{8D^4}\Big(4E^3(-6+6z-z^2)+8E^2(9-8z^2+4z^3-z^4) \\
&\quad + E(-72-72z+68z^2+16z^3-14z^4+2z^5-z^6)+6(4+8z-4z^3-z^4)\Big).
\end{aligned}
\tag{C-2}
$$

The numerical difficulties at $z = 0$ are handled as discussed above; the results for the absolute and relative errors for these five functions are presented in figures C-6 through C-10, respectively. It was decided to use 13 terms in the power series for $L(z)$ below radius 0.91, 13 terms for $L'(z)$ below radius 1.06, 15 terms for $L''(z)$ below radius 1.33, 23 terms for $L'''(z)$ below radius 2.15, and 25 terms for $L''''(z)$ below radius 2.38. Thus, for example, from figure C-10, the maximum relative error for $L''''(z)$ is 11.5 decimals for all complex z.

The relative error curve for $L(z)$ has a bump in the neighborhood of radius 1; this is due to a zero of $L(z)$ at $z = -0.9289$. Also, $L'''(z)$ has a zero at $z = 3.9977$, while $L''''(z)$ has a zero at $z = 7.6874$; however, the plots did not have to be carried out that far to make the choices for the switch radius.

For $k = 3$, that is, for a chi-squared RV with $2k + 2 = 8$ DOF, the function $L(z)$ and its first four derivatives were given in equation (40), namely

$$L(z) = \log\left(\frac{1-\exp(-z)\,(1+z+z^2/2+z^3/6)}{z^4}\right) \quad \text{for } z \neq 0, \quad L(0) = -\log(24),$$

$$L'(z) = -\frac{4}{z}+\frac{z^3}{6D}, \quad E \equiv \exp(z), \quad D \equiv E-(1+z+z^2/2+z^3/6),$$

$$L''(z) = \frac{4}{z^2}+\frac{z^2}{12\,D^2}\left(E\,(6-2z)-(6+4z+z^2)\right),$$

$$\begin{aligned} L'''(z) = {} & -\frac{8}{z^3}+\frac{z}{36\,D^3}\Big(6E^2\,(6-6z+z^2)+E\,(-72+24z^2+12z^3-3z^4+z^5) \\ & +36+36z+6z^2-12z^3-6z^4-z^5\Big), \end{aligned} \qquad \text{(C-3)}$$

$$\begin{aligned} L''''(z) = {} & \frac{1}{216\,D^4}\Big(36E^3\,(6-18z+9z^2-z^3)+24E^2\,(-27+54z+27z^2-15z^4+6z^5-z^6) \\ & +E\,(648-648z-1620z^2-1188z^3+126z^4+162z^5+3z^6-21z^7+3z^8-z^9) \\ & -3(72-216z^2-336z^3-186z^4-24z^5+20z^6+8z^7+z^8)\Big)+\frac{24}{z^4}. \end{aligned}$$

The numerical difficulties at $z = 0$ are handled as discussed above; the results for the absolute and relative errors for these five functions are presented in figures C-11 through C-15, respectively. It was decided to use 15 terms in the power series for $L(z)$ below radius 1.45, 16 terms for $L'(z)$ below radius 1.66, 17 terms for $L''(z)$ below radius 1.88, 22 terms for $L'''(z)$ below radius 2.58, and 23 terms for $L''''(z)$ below radius 2.90. Thus, for example, from figure C-15, the maximum relative error for $L''''(z)$ is 10.9 decimals for all complex z.

MATLAB programs for all the functions in equations (C-1) through (C-3) are available from the author upon request.

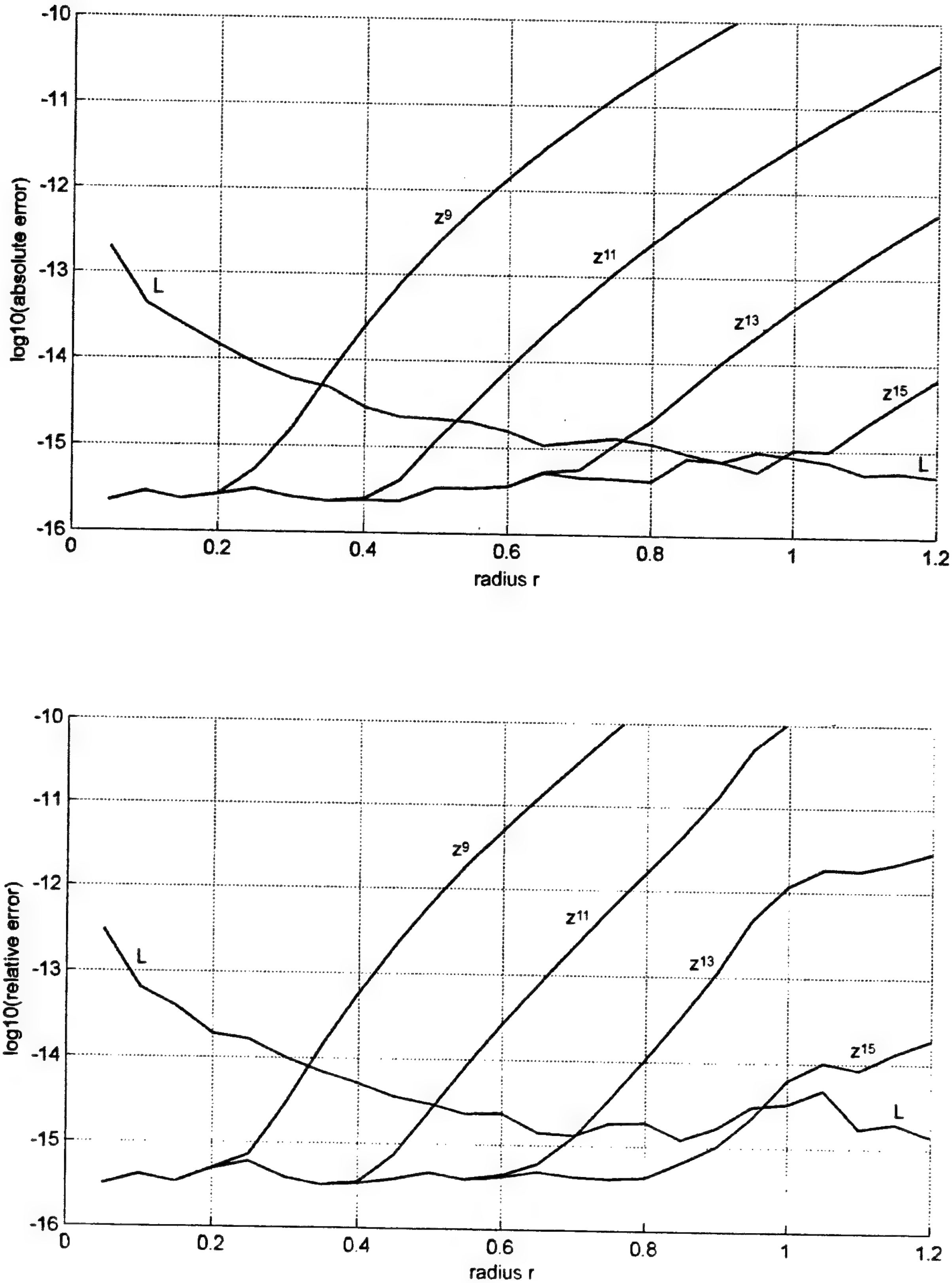


Figure C-1. Errors for L(z) in Equation (C-1)

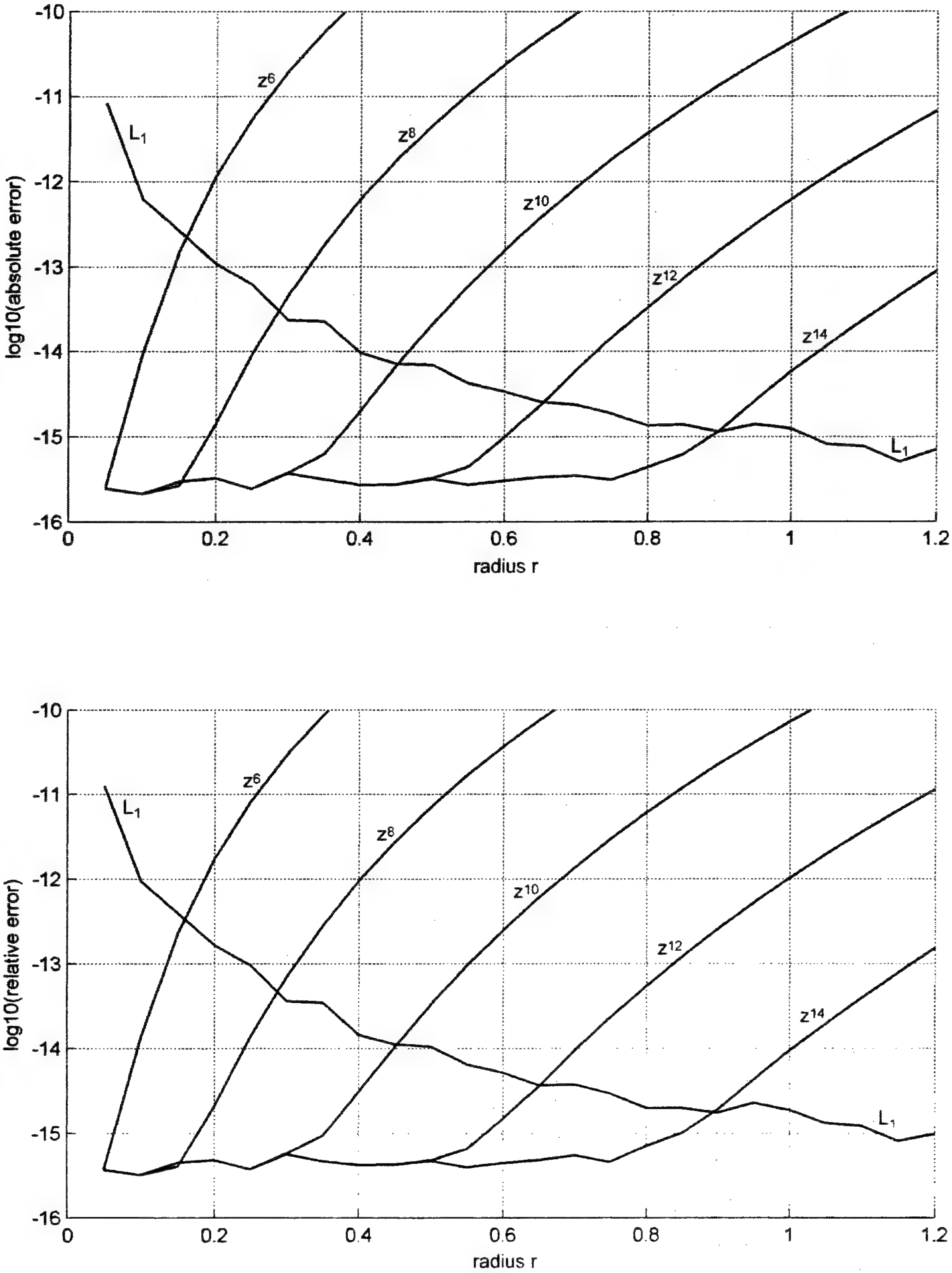


Figure C-2. Errors for L'(z) in Equation (C-1)

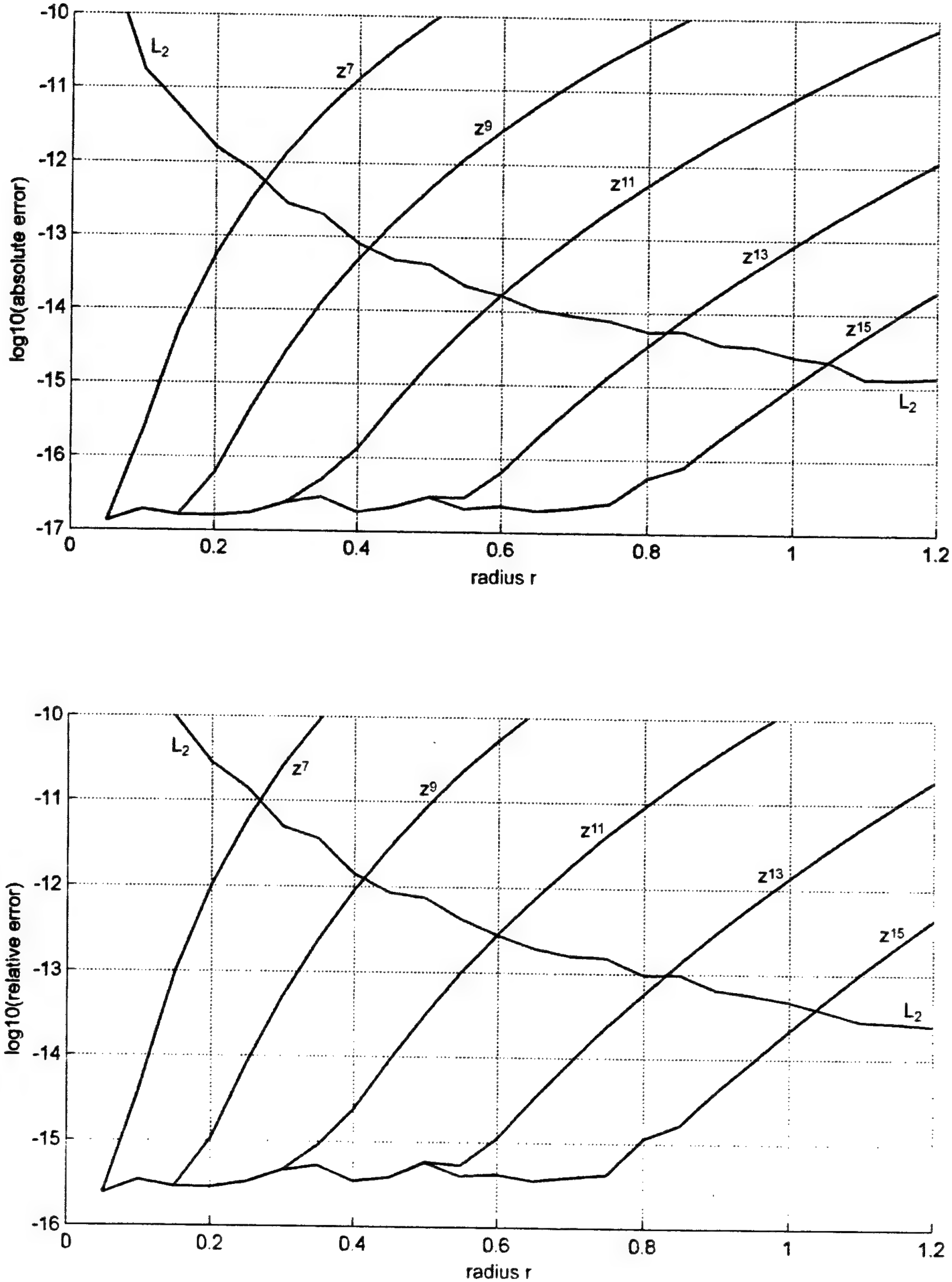


Figure C-3. Errors for L''(z) in Equation (C-1)

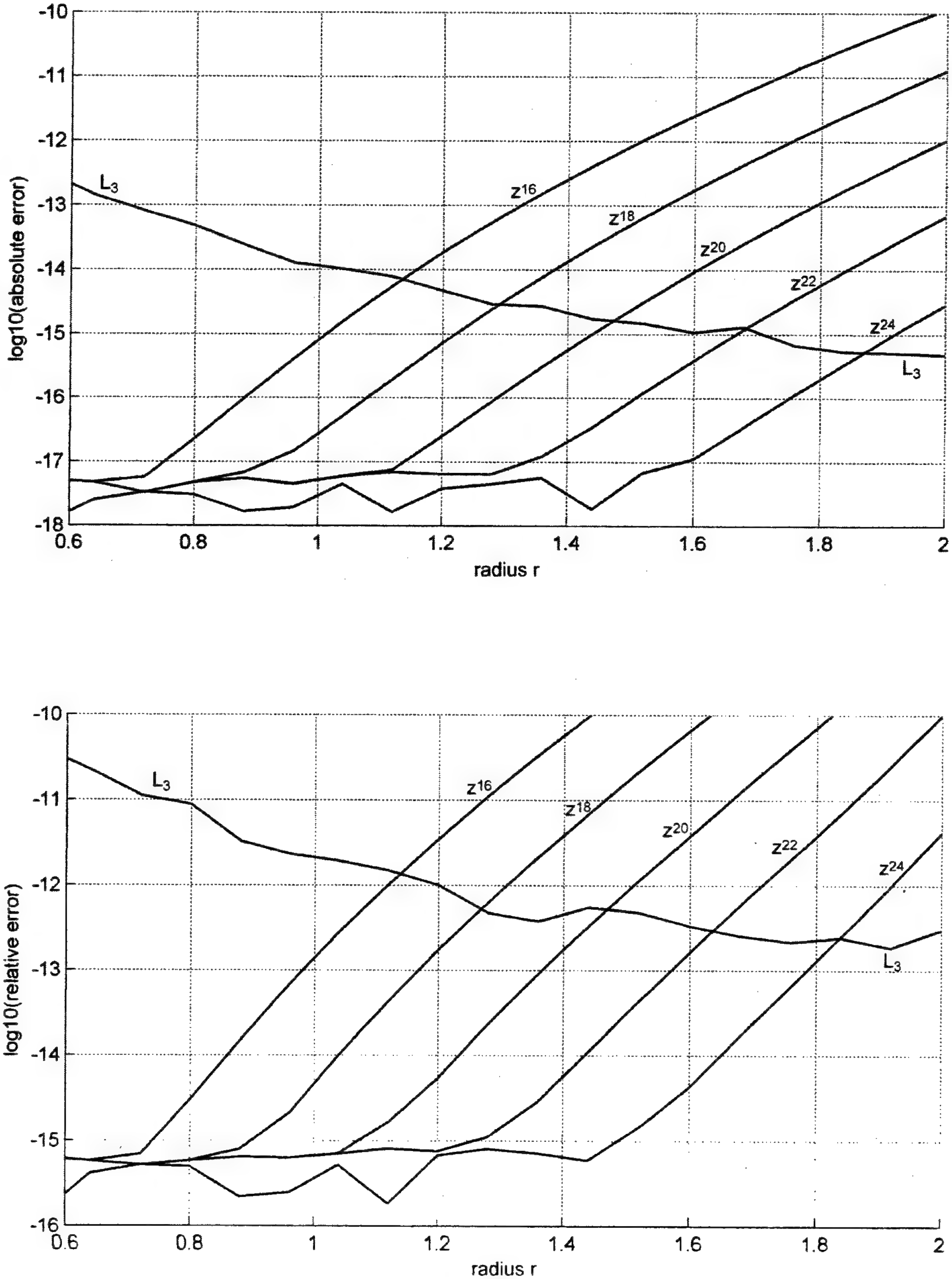
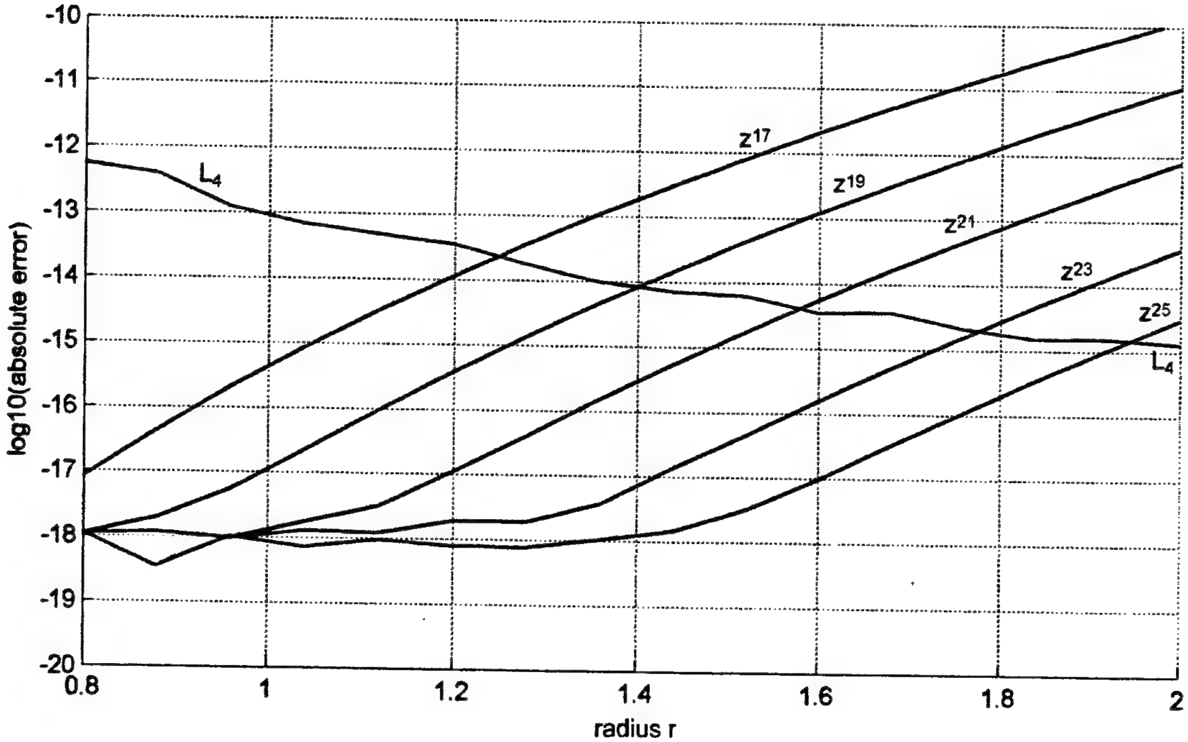


Figure C-4. Errors for $L'''(z)$ in Equation (C-1)



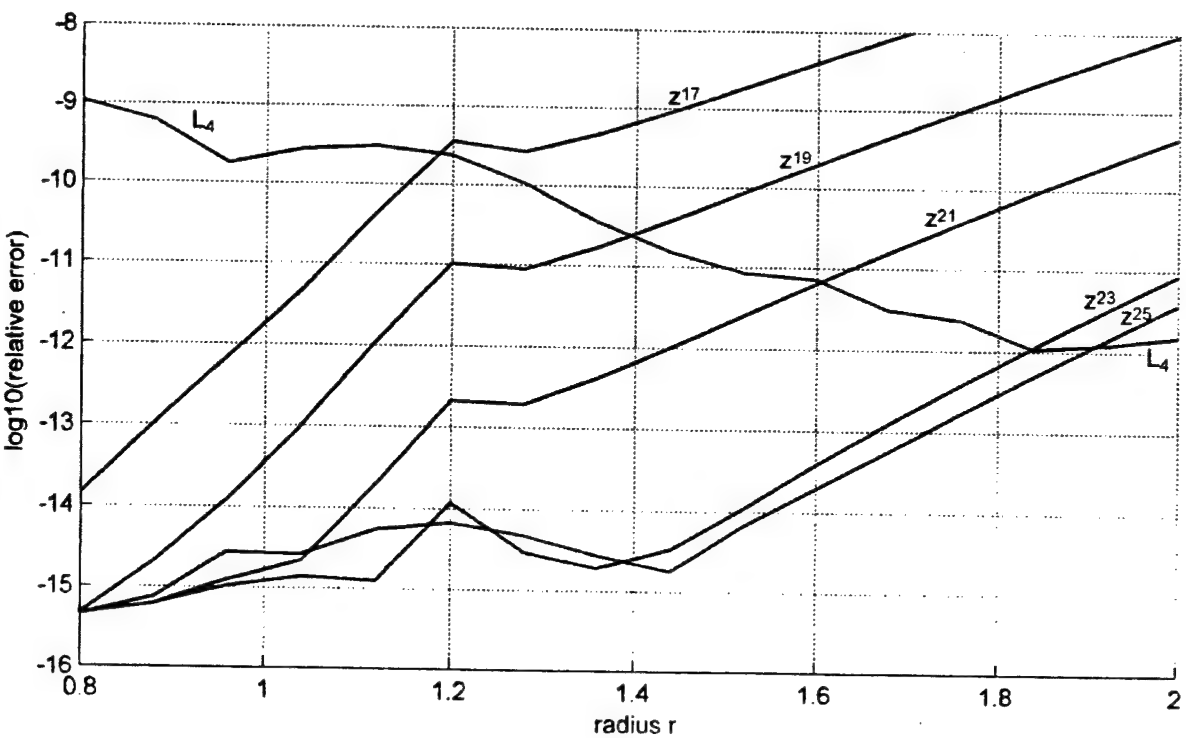
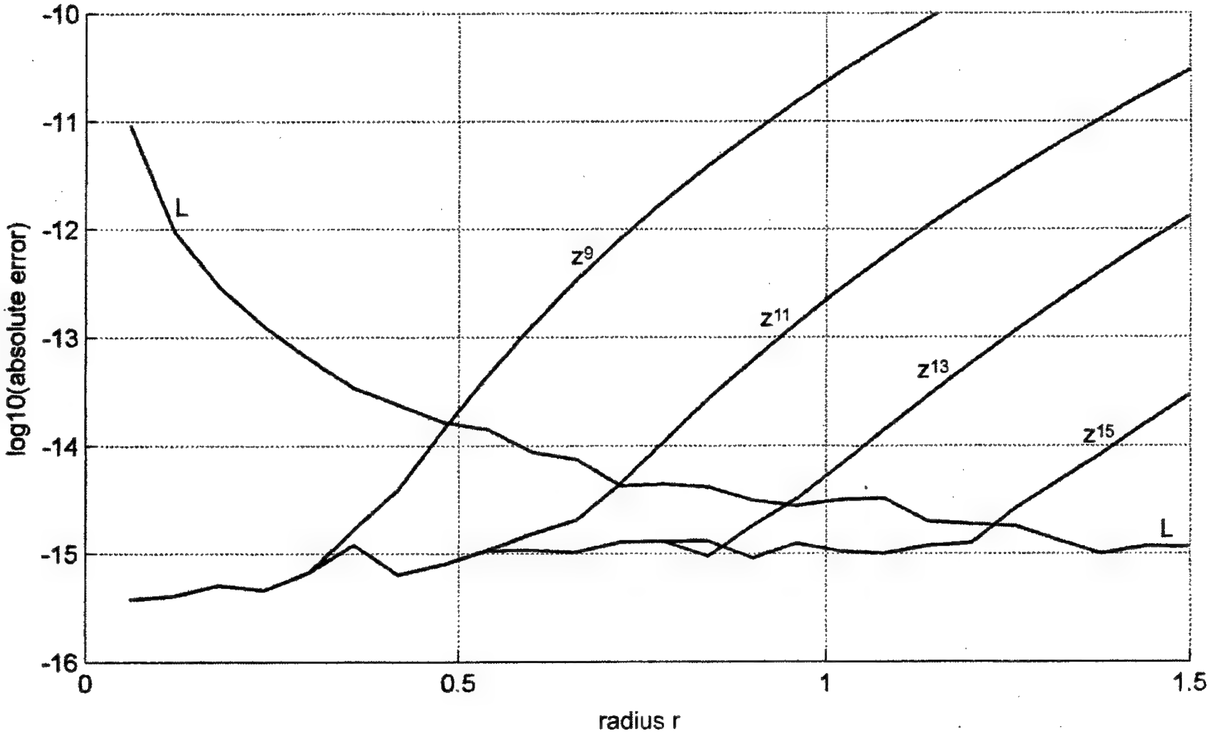


Figure C-5. Errors for L""(z) in Equation (C-1)



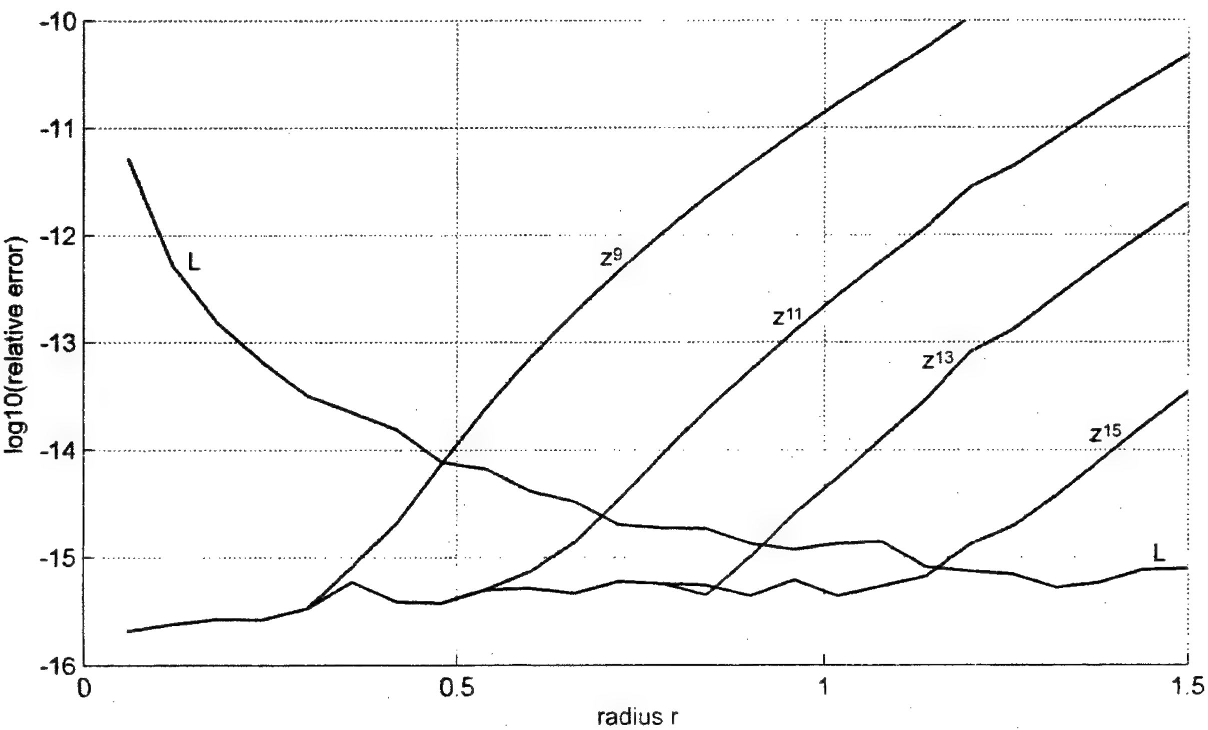
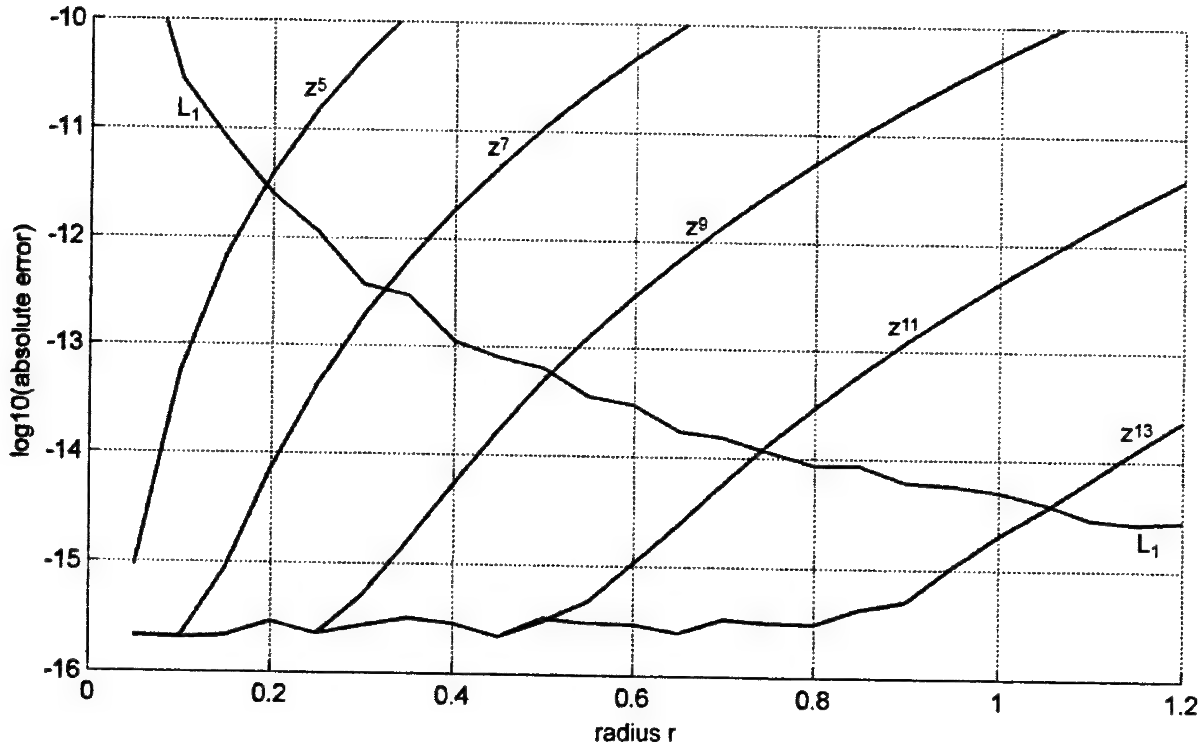


Figure C-6. Errors for L(z) in Equation (C-2)



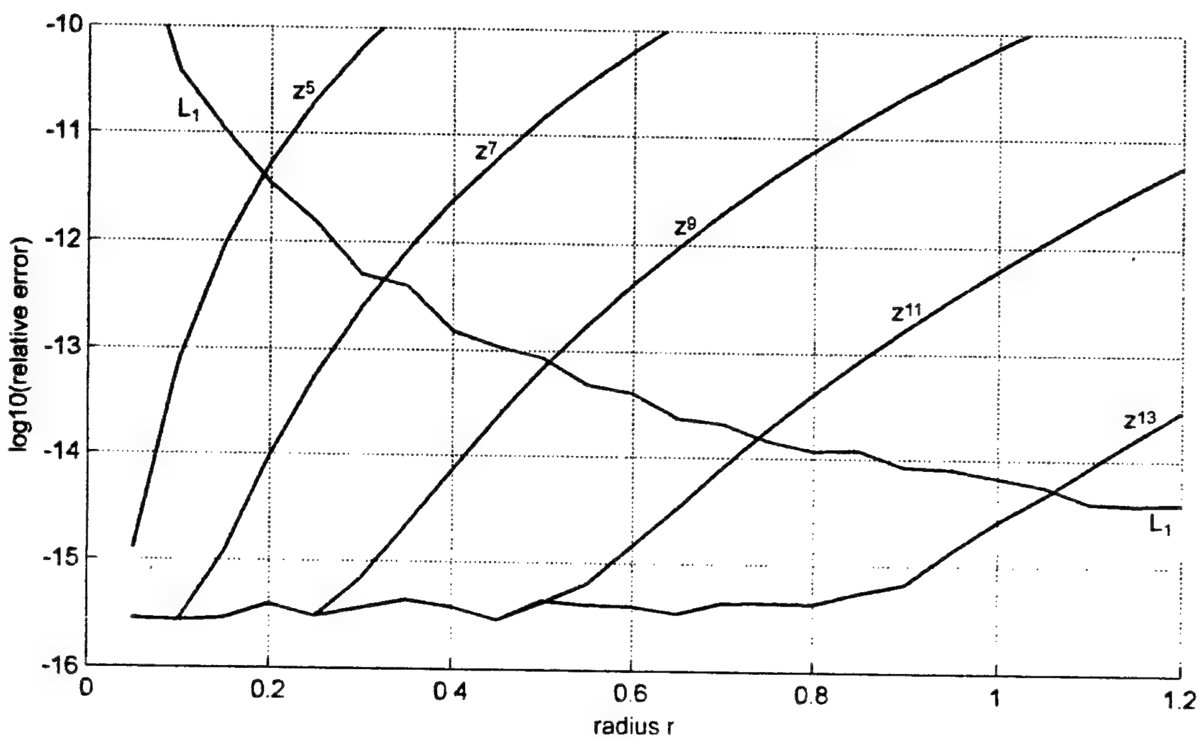
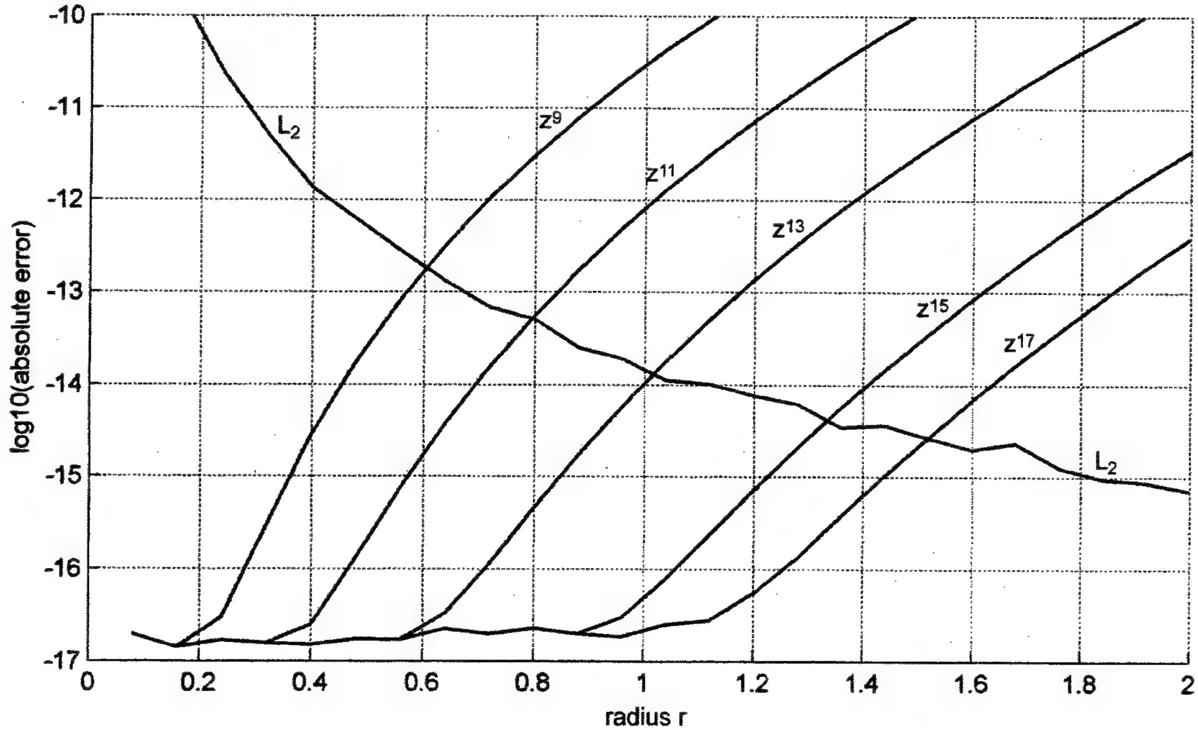


Figure C-7. Errors for L'(z) in Equation (C-2)



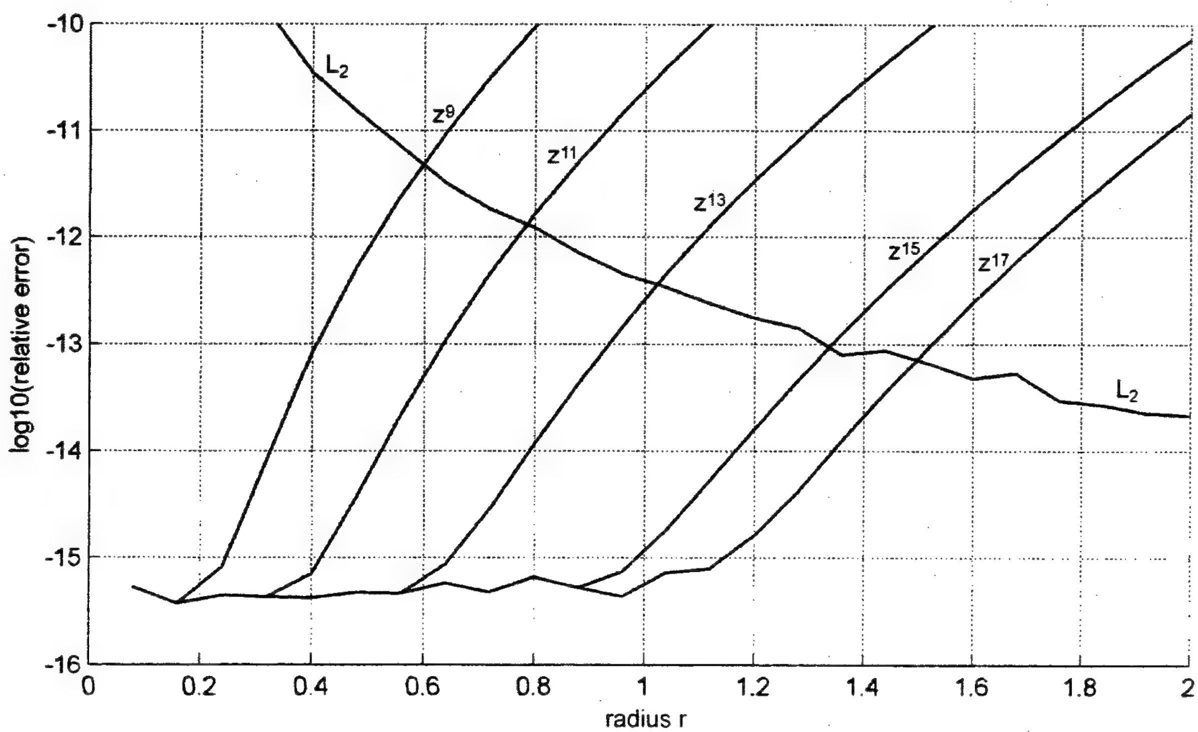


Figure C-8. Errors for L"(z) in Equation (C-2)

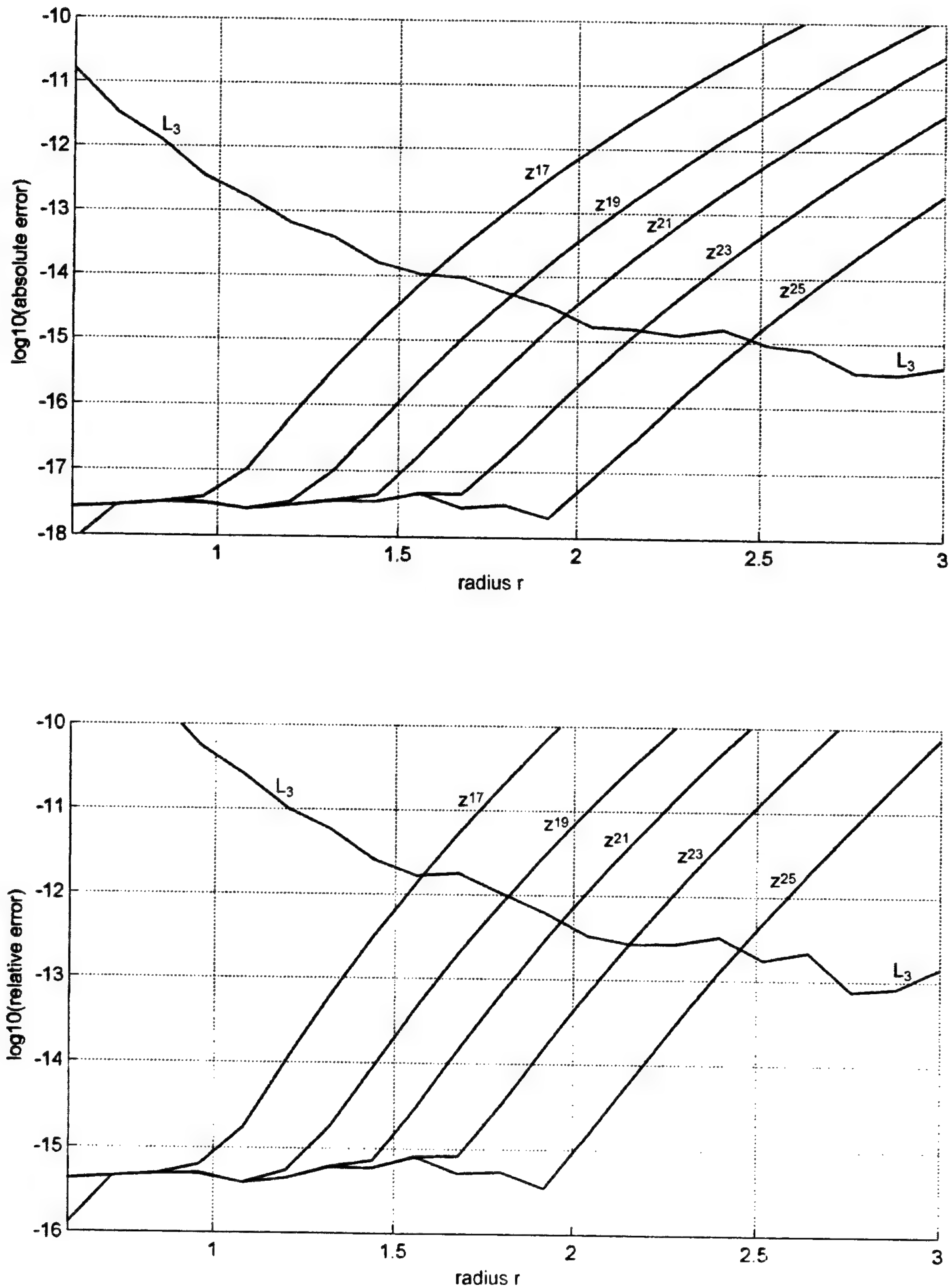


Figure C-9. Errors for L'''(z) in Equation (C-2)

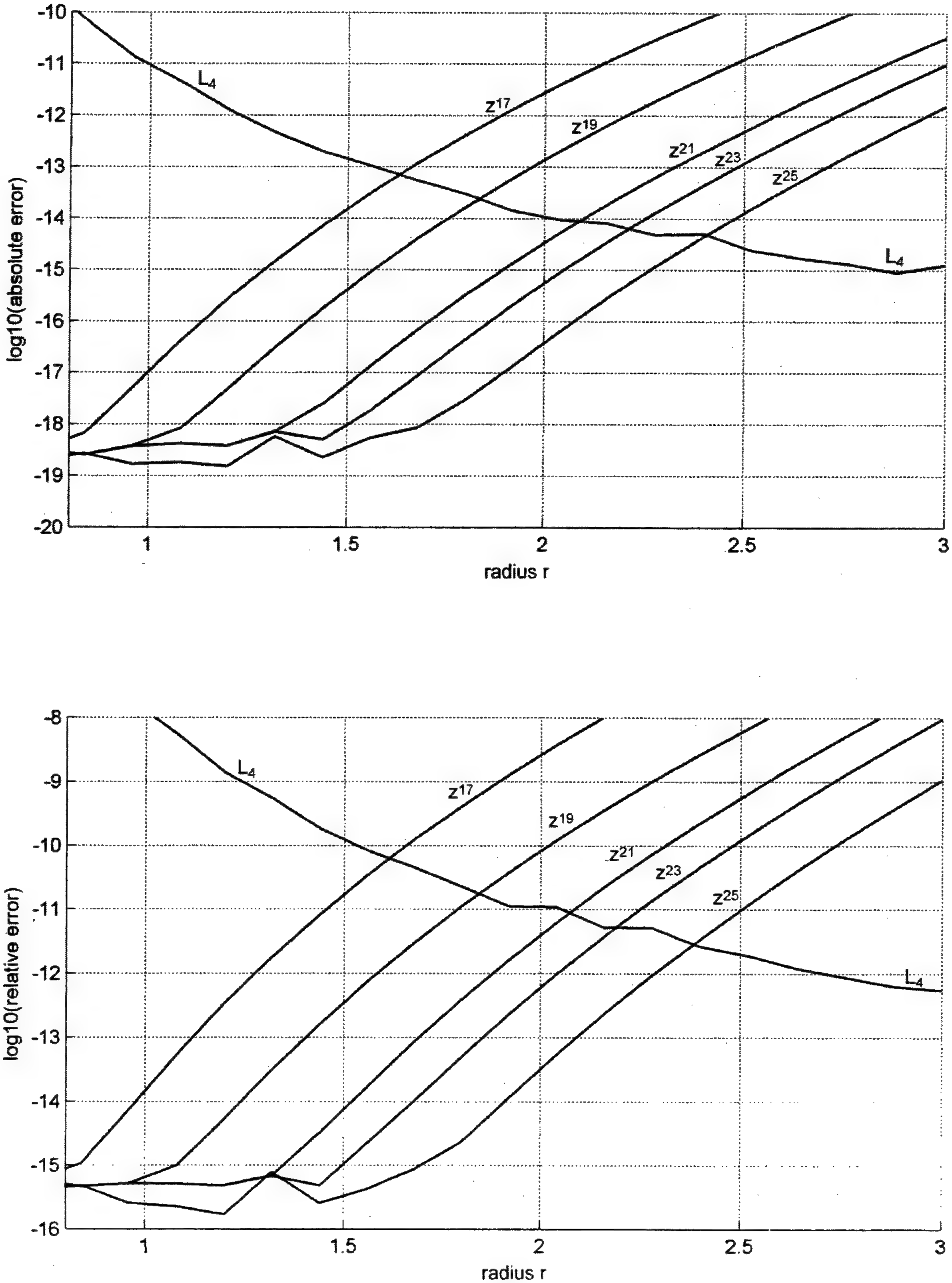
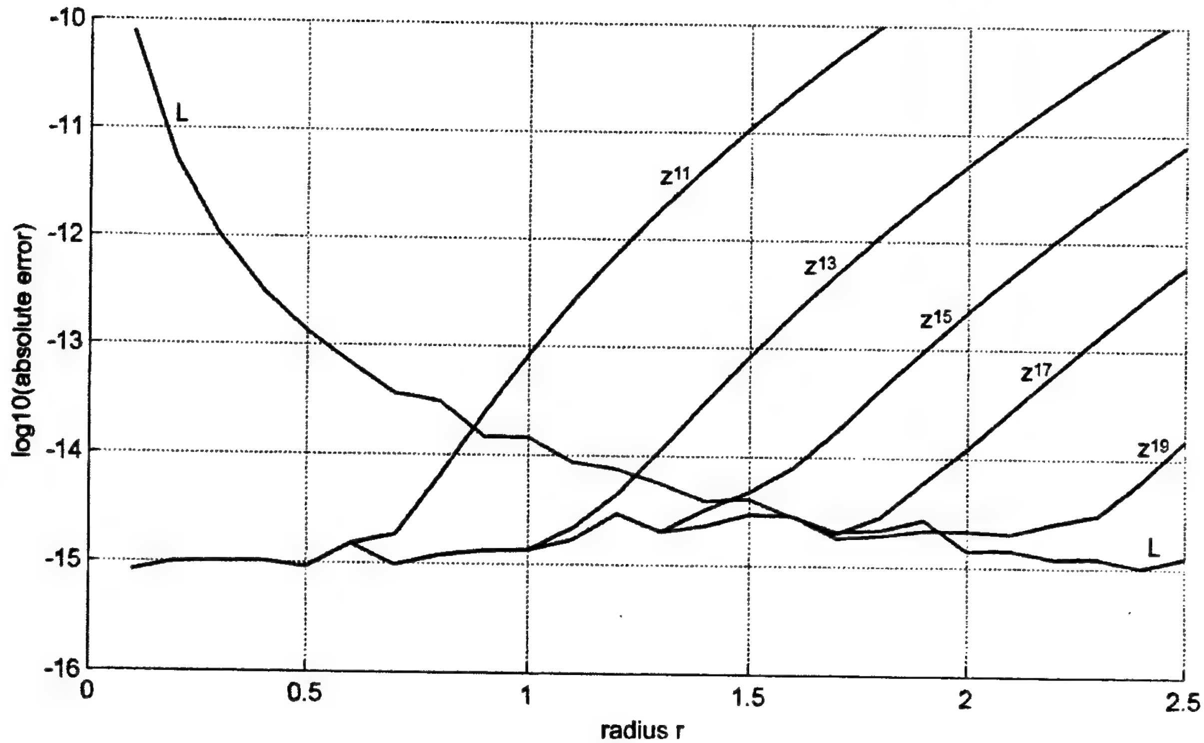


Figure C-10. Errors for L""(z) in Equation (C-2)



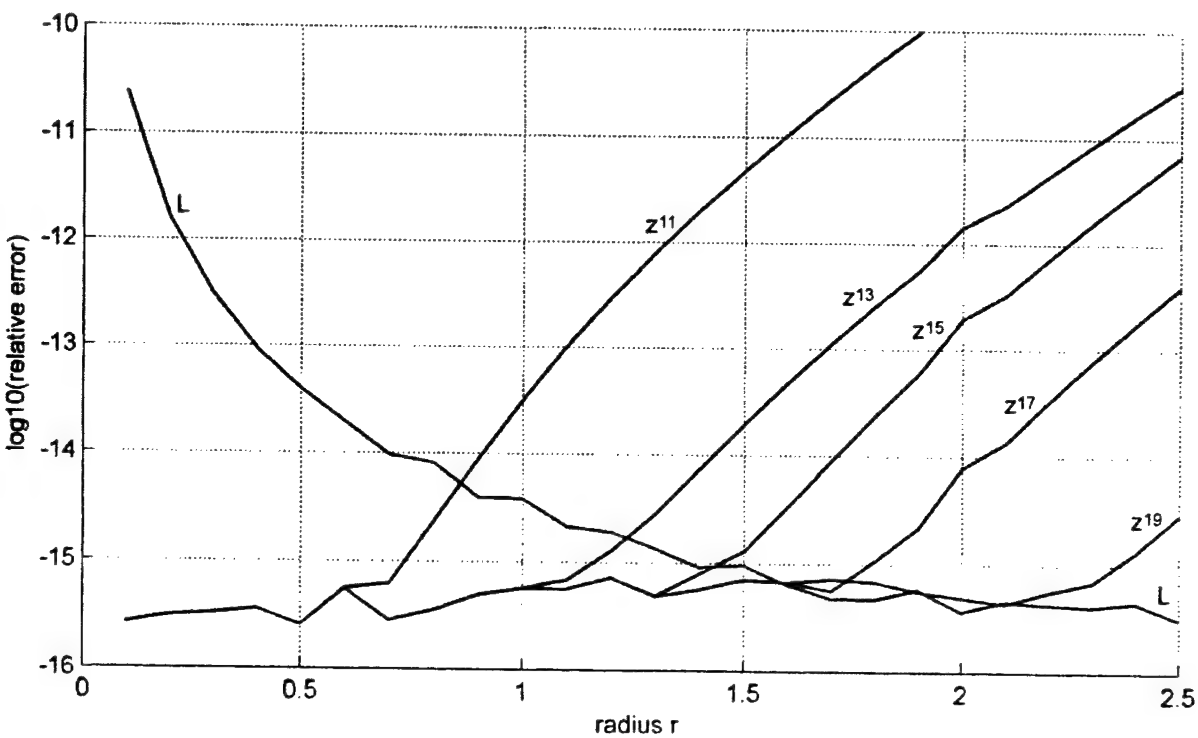


Figure C-11. Errors for L(z) in Equation (C-3)

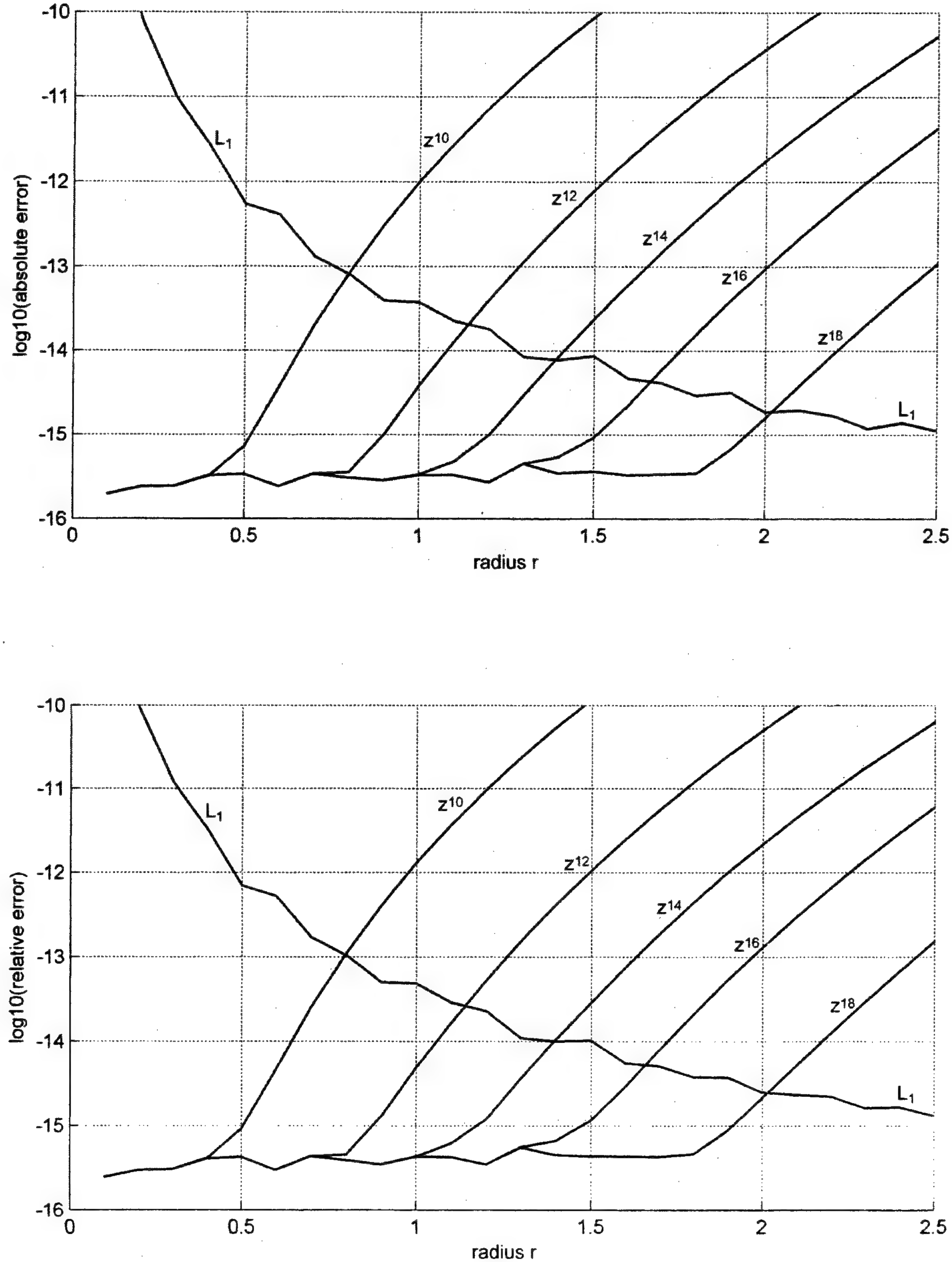


Figure C-12. Errors for L'(z) in Equation (C-3)

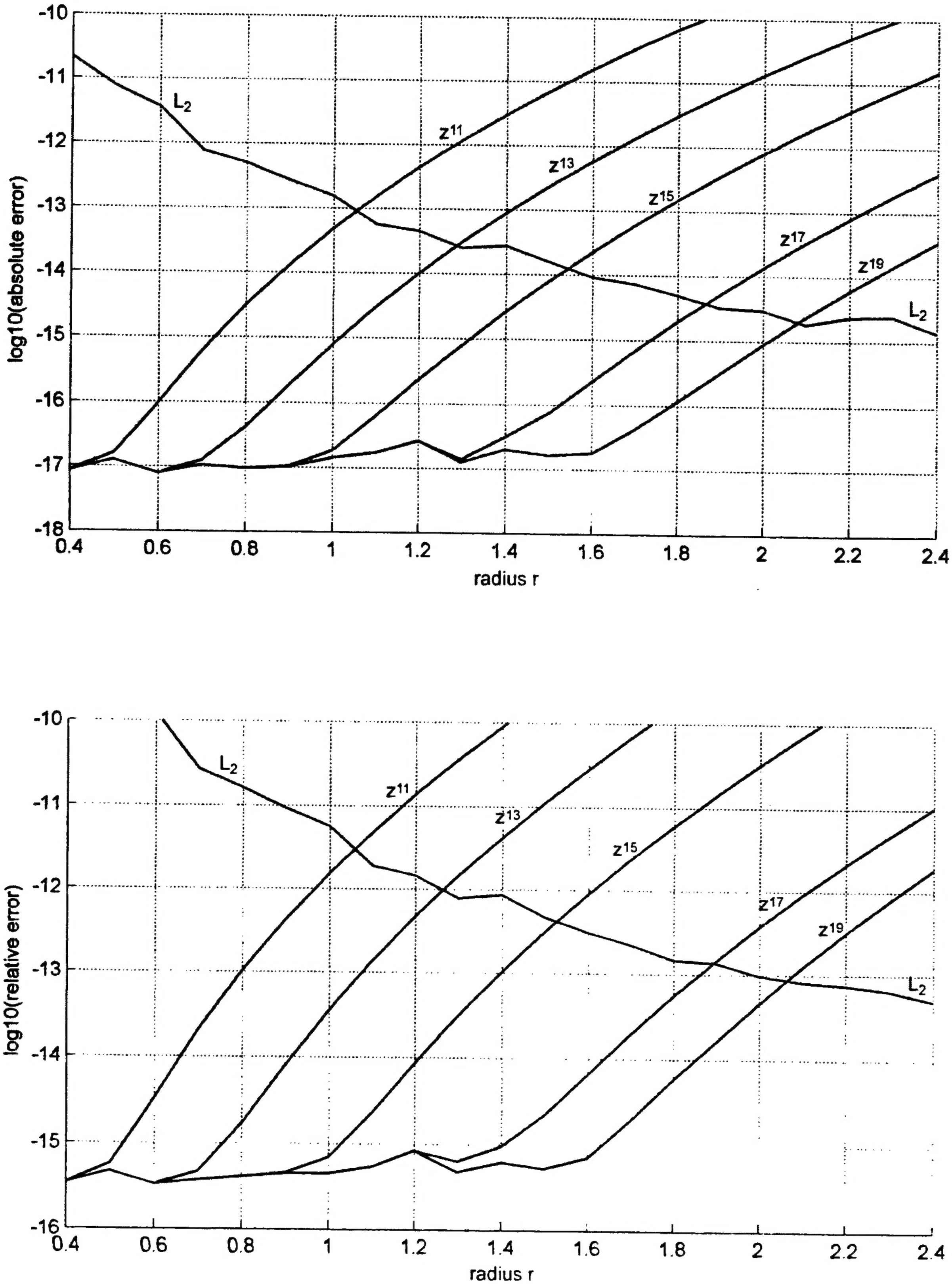
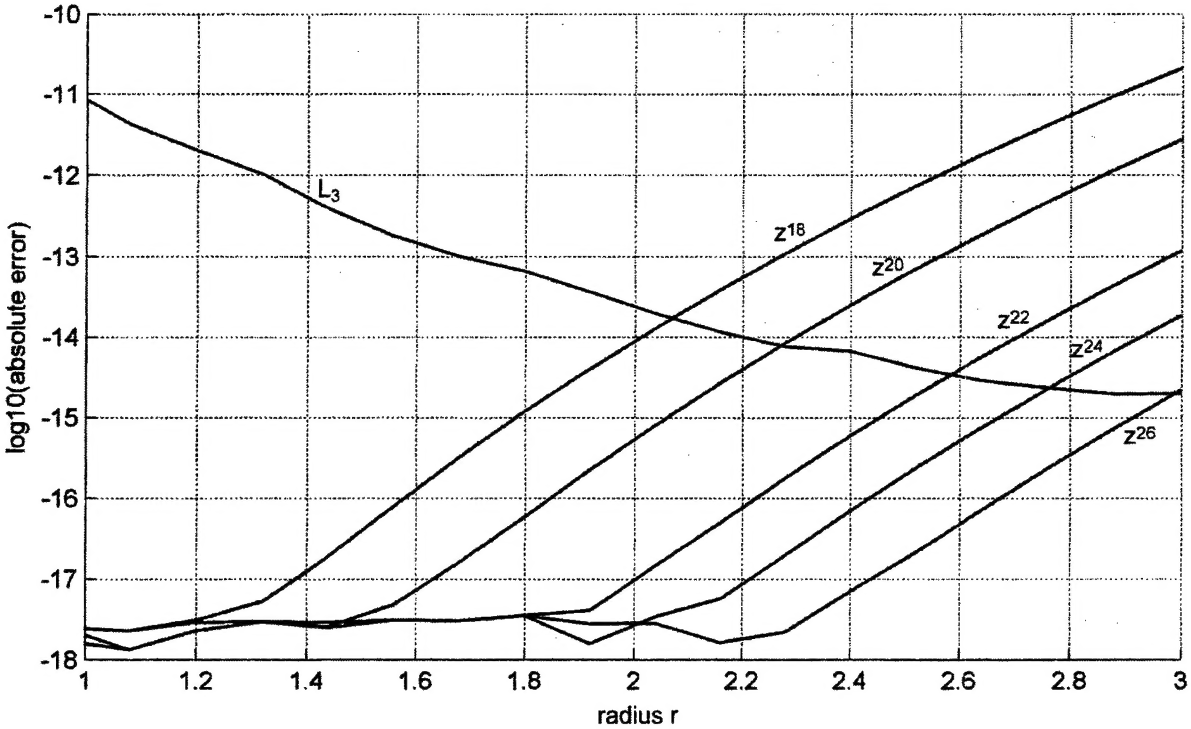


Figure C-13. Errors for L"(z) in Equation (C-3)



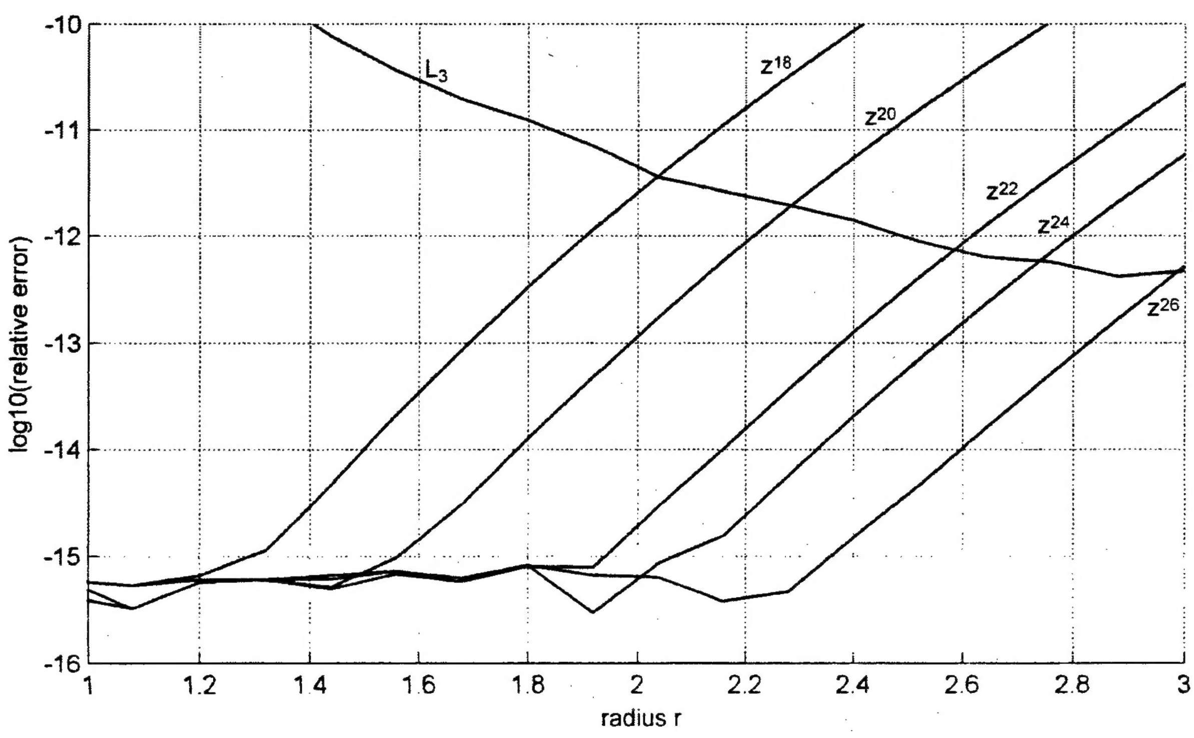
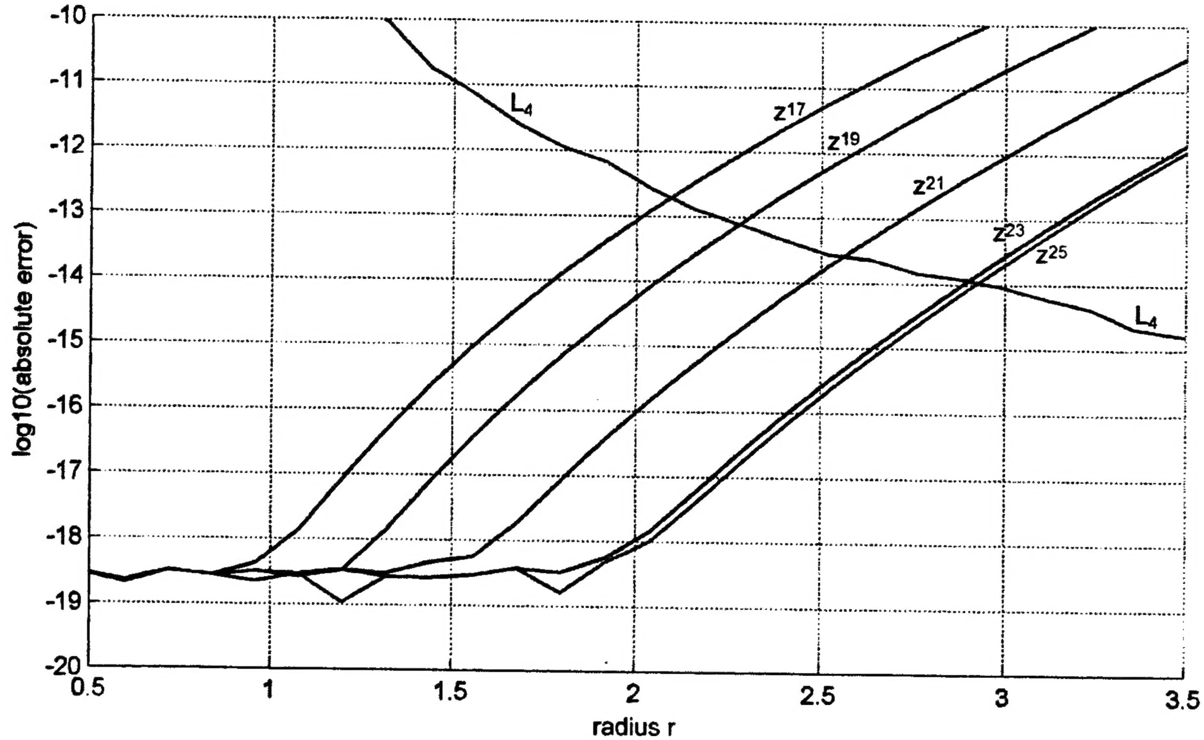


Figure C-14. Errors for L'''(z) in Equation (C-3)



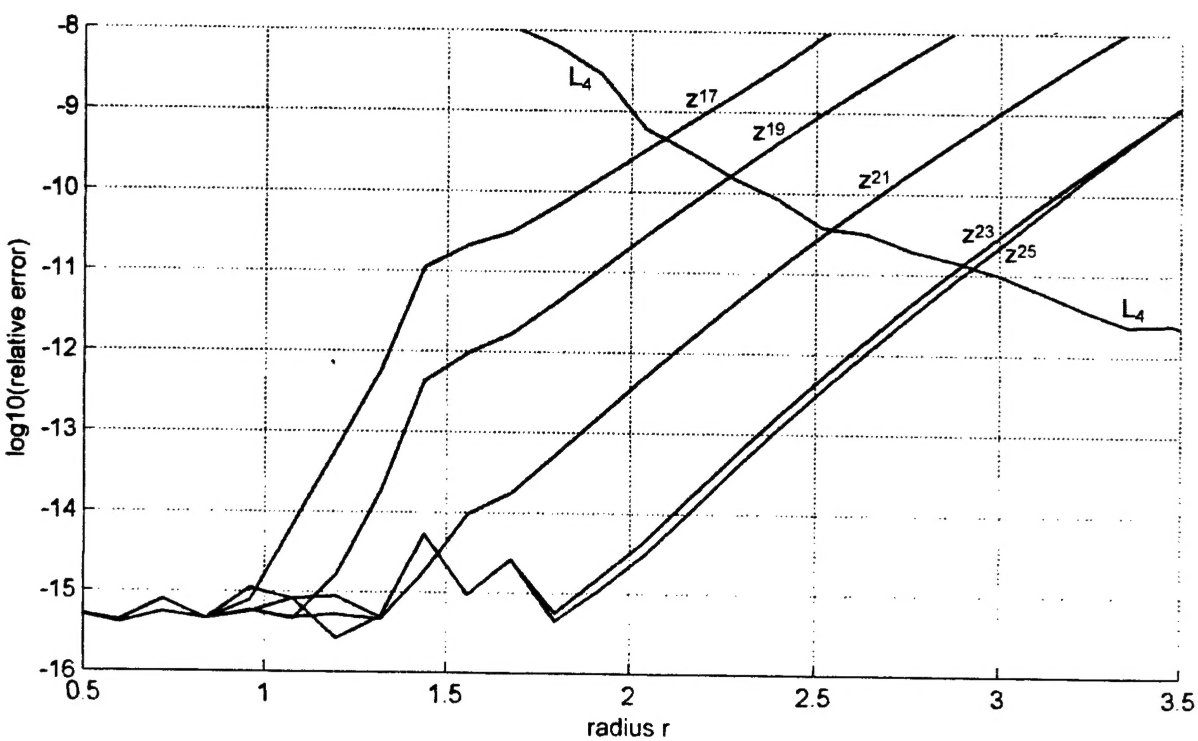


Figure C-15. Errors for L""(z) in Equation (C-3)

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